Supplemental Material for "High-Temperature Quantum Oscillations of a Non-equilibrium Non-Fermi Liquid"

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Appendix A: Floquet Boltzmann equation

In this section we derive the scattering rates entering Eq.(3) of the main text. Our goal is to show that they take the form of a Floquet generalization of Fermi's Golden Rule, Eq.(4), and to clarify the physical meaning of the different terms.

$$\langle W_t^{\alpha\to\beta}\rangle = \frac{2\pi}{\hbar}\sum_{n,q}|V^n_{\alpha\to\beta}(q)|^2\delta(\epsilon^F_\beta-\epsilon^F_\alpha+\hbar\omega_q+n\hbar\Omega) \tag{A-1}$$

This has the structure of a Fermi Golden Rule, but with Floquet-assisted processes where energy conservation holds modulo multiples of the drive frequency. Unlike equilibrium Fermi Golden Rule, these rates do not satisfy detailed balance.

We start the derivation from the emission part of the collision integral for the one-body electron density matrix evolution $\partial_t\hat\rho_t + (i/\hbar)[\hat H_{\rm e}(t),\hat\rho_t] = (1/\hbar)J_e[t,\hat\rho_t] + (1/\hbar)J_a[t,\hat\rho_t]$ (see Refs.[3, 5]):

$$J_e[t,\hat\rho_t] = \frac{1}{\hbar}\sum_q\int_{t_0}^t dt'(N_q+1)e^{-i\omega_q(t-t')}\left[U_{\rm e}(t,t')(1-\hat\rho_{t'})\chi_q^\dagger\hat\rho_{t'}U_{\rm e}(t',t),\chi_q\right] + h.c., \tag{A-2}$$

where $id_t\hat U_{\rm e}(t,t') = \hat H_{\rm e}(t)\hat U_{\rm e}(t,t')$ is the evolution operator of the electronic subsystem, $[\hat A,\hat B]\equiv\hat A\hat B-\hat B\hat A$ is the standard commutator and the absorption contribution follows from the replacements $N_q+1\to N_q,\ \omega_q\to-\omega_q,\ \hat\chi_q\to\hat\chi_q^\dagger$ [5], with q above to be a label for a boson quantum state.

Due to the bath coupling we expect a drive-bath stabilized steady state of the system is reached in the form of the periodic Gibbs ensemble form introduced in Eq.(2) of the main text (see also Refs. [6–9]). Below we show that indeed there is periodic Gibbs ensemble which is a self-consistent solution of the kinetic equations in the weak coupling limit between the system and the bath. By expanding the evolution operator in the Floquet basis as:

$$U_{\rm e}(t,t') = \sum_{\alpha,n,m}e^{-i\epsilon^F_\alpha(t-t')}e^{-in\Omega t}e^{im\Omega t'}|\phi^n_\alpha\rangle\langle\phi^m_\alpha|. \tag{A-3}$$

Projecting the kinetic equation onto the diagonal Floquet components and performing standard manipulations (see Ref. [5]), we obtain the steady state equation as:

$$0 = \frac{1}{\hbar}\sum_{nm}\langle\phi^n_\alpha|\,(J_e+J_a)\,|\phi^m_\alpha\rangle\,e^{i(n-m)\Omega t} = \frac{1}{\hbar}\mathcal{I}_{e,\alpha}+\frac{1}{\hbar}\mathcal{I}_{a,\alpha}. \tag{A-4}$$

which, as we will see shortly is equivalent to Eq.(3) of the main text.

Due to non-triviality of the next step, we will evaluate the collision part of the Eq. (A-4) step by step starting from writing it explicitly:

$$\begin{aligned}\frac{1}{\hbar}\mathcal{I}_{e,\alpha}(t) = \frac{1}{\hbar}\sum_{nm}\langle\phi^n_\alpha|\,J_e[t,\rho]\,|\phi^m_\alpha\rangle\,e^{i(n-m)\Omega t} = \frac{1}{\hbar^2}\sum_{l_1l_2l_3l_4}\sum_q\int_{t_0}^t dt'(N_q+1)e^{-il_1\Omega t+il_2\Omega t'-il_3\Omega t'+il_4\Omega t}\times\\ \times\Bigg[\sum_{\beta'}e^{-i(\hbar\omega_q+\epsilon^F_\alpha-\epsilon^F_{\beta'})(t-t')/\hbar}(1-f_\alpha)f_{\beta'}\,\langle\phi^{l_2}_\alpha|\,\hat\chi_q^\dagger\,|\phi^{l_3}_{\beta'}\rangle\,\langle\phi^{l_4}_{\beta'}|\,\hat\chi_q\,|\phi^{l_1}_\alpha\rangle\\ -\sum_\beta e^{-i(\hbar\omega_q+\epsilon^F_\beta-\epsilon^F_\alpha)(t-t')/\hbar}(1-f_\beta)f_\alpha\,\langle\phi^{l_4}_\alpha|\,\hat\chi_q\,|\phi^{l_1}_\beta\rangle\,\langle\phi^{l_2}_\beta|\,\hat\chi_q^\dagger\,|\phi^{l_3}_\alpha\rangle\Bigg] + h.c.\end{aligned} \tag{A-5}$$

where we used $\delta_{\alpha\beta} = \sum_{l_1 l_2} e^{i(l_1-l_2)\Omega t}\langle\phi_\alpha^{l_1}|\phi_\beta^{l_2}\rangle$. It is useful to introduce $\chi_{q,\alpha\beta}(\ell) \equiv \sum_{l_0} \langle\phi_\alpha^{l_0}|\,\hat{\chi}_q\,|\phi_\beta^{l_0+\ell}\rangle$ relabel modes as $l_1 = l_4 + \ell,\ l_2 = l_3 + \ell'$ and relabel dummy parameter $t' - t = \tau$ to cast the above as:

$$\frac{1}{\hbar}\mathcal{I}_{e,\alpha}(t) = \frac{1}{\hbar}\sum_{nm}\langle\phi_\alpha^n|\,J_e[t,\rho]\,|\phi_\alpha^m\rangle\, e^{i(n-m)\Omega t} = \frac{1}{\hbar^2}\sum_{\ell\ell'} e^{-i\ell\Omega t + i\ell'\Omega t}\sum_{q,\beta}\int_{t_0-t}^{0} d\tau (N_q+1)\times$$
$$\times\left[e^{i(\hbar\omega_q + \epsilon_\alpha^F - \epsilon_\beta^F + \ell'\hbar\Omega)\tau/\hbar}\bar{f}_\alpha f_\beta[\chi_{q,\beta\alpha}(\ell')]^\dagger\chi_{q,\beta\alpha}(\ell) - (\alpha\leftrightarrow\beta)\right] + h.c. \quad \text{(A-6)}$$

After we employ the standard approximation $\int_{t_0}^0 d\tau f(\tau) = \lim_{\eta\to 0}\int_{-\infty}^0 d\tau e^{\eta\tau/\hbar} f(\tau)$ we obtain the generic time dependent collision integral as:

$$\frac{1}{\hbar}\mathcal{I}_{e,\alpha}(t) = \frac{1}{\hbar}\sum_{\ell\ell'} e^{-i\ell\Omega t + i\ell'\Omega t}\sum_{q,\beta}(N_q+1)\left[\frac{\bar{f}_\alpha f_\beta[\chi_{q,\beta\alpha}(\ell')]^\dagger\chi_{q,\beta\alpha}(\ell)}{i(\hbar\omega_q + \epsilon_\alpha^F - \epsilon_\beta^F + \ell'\hbar\Omega) + \eta} - (\alpha\leftrightarrow\beta)\right] + h.c. \quad \text{(A-7)}$$

At this stage, the collision integral remains explicitly time dependent through harmonics of the drive. Since the steady-state occupations f_α are time independent, the relevant kinetic equation is obtained from its static component, i.e. the average over one drive period, which corresponds to $\ell = \ell'$. We then simplify this component using $\frac{1}{ix+\eta} + \frac{1}{-ix+\eta} = \frac{2\eta}{x^2+\eta^2} \to 2\pi\delta(x)$, which yields the emission part of the Floquet–Boltzmann collision integral in the form

$$\frac{1}{\hbar}\bar{\mathcal{I}}_{e,\alpha} = \sum_\beta\left[\bar{f}_\alpha f_\beta\langle W_{t,e}^{\beta\to\alpha}\rangle_T - (\beta\leftrightarrow\alpha)\right] \quad \text{(A-8)}$$

where $\langle W_{t,e}^{\alpha\to\beta}\rangle_T = (2\pi/\hbar)\sum_q(N_q+1)\sum_\ell|\chi_{q,\alpha\beta}(\ell)|^2\delta(\hbar\omega_q + \epsilon_\beta^F - \epsilon_\alpha^F + \ell\hbar\Omega)$. The derived result combined with the absorption part is the Eq.(4) of the main text ($W_t^{\beta\to\alpha} \equiv W_{t,e}^{\beta\to\alpha} + W_{t,a}^{\beta\to\alpha}$), has the structure of a Fermi Golden Rule, but generalized to periodically driven systems. Eq.(3) is obtained from the stationary condition as in Eq.(A-4).

Importantly:

- The integer n labels the exchange of n quanta of the drive with frequency Ω.
- Energy conservation is modified to $\epsilon_\beta^F - \epsilon_\alpha^F + \hbar\omega_q + n\hbar\Omega = 0$ (Floquet umklapp processes).
- Absence of detailed balance. This ultimately arises because the Floquet harmonics of a state α are not determined exclusively by its Floquet energy ϵ_α^F. As a result, the key condition needed to establish detailed balance is not satisfied.
- The matrix element $\langle W_t^{\alpha\to\beta}\rangle_T$ contains overlaps between different Floquet harmonics.

In the limit of vanishing drive amplitude, only the $n = 0$ term survives and the expression reduces to the standard equilibrium Fermi Golden Rule. A crucial difference with equilibrium is that the rates do not satisfy detailed balance. As a result, the steady state occupations f_α are not constrained to take a Fermi–Dirac form. This lack of detailed balance is ultimately responsible for the non-equilibrium steady states discussed in the main text, including the emergence of non-analytic features in the distribution function.

Appendix B: Floquet spectrum of periodically driven Landau levels

Here we discuss the details of the kinetics of driven Landau levels. First, we rewrite the Hamiltonian in Eq.(6) in its operator form as:

$$\hat{h}_t = \left[\hat{\mathbf{p}} - \frac{e}{\hbar}\mathbf{A}_0 - \frac{e}{\hbar}\mathbf{A}(t)\right]^2/(2m), = \hbar\omega_c\left(1/2 + \hat{a}^\dagger\hat{a}\right) - \hat{a}z_t^* - \hat{a}^\dagger z_t + c_t, \quad \text{(B-1)}$$

where

$$z_t = \sqrt{\frac{\hbar\omega_c}{2m}}(A^x(t) - iA^y(t)) = z_+e^{-i\Omega t} + z_-e^{i\Omega t}, \qquad \mathbf{A}(t) = -\frac{i}{\Omega}\boldsymbol{\mathcal{E}}_+\exp(-i\Omega t) + \frac{i}{\Omega}\boldsymbol{\mathcal{E}}_-\exp(i\Omega t), \quad \text{(B-2)}$$

$$\mathbf{B} = \nabla\times A_0, \qquad a^\dagger|N\rangle = \sqrt{N+1}\,|N+1\rangle, \qquad a|N\rangle = \sqrt{N}\,|N-1\rangle, \quad \text{(B-3)}$$

$$z_+ = -\frac{i}{\Omega}\sqrt{\frac{\hbar\omega_c}{2m}}(\mathcal{E}_+^x - i\mathcal{E}_+^y), \qquad z_- = \frac{i}{\Omega}\sqrt{\frac{\hbar\omega_c}{2m}}(\mathcal{E}_-^x - i\mathcal{E}_-^y), \qquad c_t = \frac{e^2\mathbf{A}(t)^2}{2m\hbar^2}. \quad \text{(B-4)}$$

We proceed by performing the unitary displacement transformation $D = \exp[\alpha a^\dagger - \alpha^* a]$ with $D^\dagger = D^{-1}$ [21]. In order to explicitly show the displacement transformation, we will employ two useful identities. First is the special form of the Baker–Campbell–Hausdorff formula known as Campbell identity, which explicitly reads as

$$e^X Y e^{-X} = Y + [X, Y] + \frac{1}{2!}[X, [X, Y]] + \frac{1}{3!}[X, [X, [X, Y]]] + \dots \tag{B-5}$$

which in our case allows to write:

$$D^\dagger a D = Y + [X, Y] = a + \alpha \qquad D^\dagger a^\dagger D = Y + [X, Y] = a^\dagger + \alpha^*. \tag{B-6}$$

Further we use:

$$\frac{de^{-\beta X(t)}}{dt} = -\int_0^\beta e^{-(\beta-u)X(t)} \frac{dX(t)}{dt} e^{-uX(t)} du, \tag{B-7}$$

noting that as $d_t X$ generally does not commute with X. By proceeding and employing the Campbell identity once again we find:

$$-i\hbar D^\dagger d_t D = i\hbar \left(\frac{dX(t)}{dt} + \frac{1}{2}\left[X(t), \frac{dX(t)}{dt} \right] \right) = i\hbar \left(\dot{\alpha}^* a - \dot{\alpha} a^\dagger + \frac{1}{2}(\alpha\dot{\alpha}^* - \alpha^*\dot{\alpha}) \right). \tag{B-8}$$

By choosing:

$$\alpha(t) = \alpha_+ e^{-i\Omega t} + \alpha_- e^{i\Omega t} = \frac{1}{\hbar}\left[\frac{z_+}{\omega_c - \Omega} e^{-i\Omega t} + \frac{z_-}{\omega_c + \Omega} e^{i\Omega t} \right] \tag{B-9}$$

we arrive to the displaced Hamiltonian

$$\begin{aligned} H'_{\rm LL} &= D^\dagger H_{\rm LL} D - i\hbar D^\dagger d_t D \\ &= \hbar\omega_c \left(a^\dagger a + \frac{1}{2} \right) + \underbrace{\frac{\mathbf{A}_1(t)^2}{2m} - \frac{1}{\hbar}\left[\frac{|z_+|^2}{\omega_c - \Omega} + \frac{|z_-|^2}{\omega_c + \Omega} + \frac{\omega_c}{\omega_c^2 - \Omega^2}(z_+ z_-^* e^{-2i\Omega t} + z_+^* z_- e^{2i\Omega t}) \right]}_{\Delta(t)}, \end{aligned} \tag{B-10}$$

which is consistent with our previous study [2]. We can absorb the time dependent constant part of the Hamiltonian into a phase as $|\psi'(t)\rangle = e^{-i\Phi(t)}|\phi(t)\rangle$ with $\Phi(t) = \Phi_0 + \int_{t_0}^t \Delta(t')dt'$. Then the Floquet Landau levels kinetics is mapped onto the standard Landau Level problem, with the full set of solutions labeled by N, r, written in Eq.(7) of the main text. Notice, since $\Delta(t)$ is independent of Landau level index, the Floquet-Bolzmann equation is unaffected by such phase factor. This can be seen from the fact that in any collision term, the evolution matrix everywhere comes in pair from $t \to t'$ and $t' \to t$, thus such Δ-related phases cancel.

Appendix C: Effective cycle averaged free energy and entropy

1. Floquet Energy

Here we clarify the definition of the physical energy used in the main text. For a periodically driven non-interacting fermion system with a single-particle Hamiltonian the Floquet states and the steady state density can be written as

$$\hat{\rho}_t = \sum_\alpha f_\alpha |\psi_\alpha^F(t)\rangle \langle\psi_\alpha^F(t)|, \qquad |\psi_\alpha^F(t)\rangle = e^{-i\epsilon_\alpha^F t/\hbar} \sum_{n=-\infty}^{\infty} e^{-in\Omega t}|\phi_\alpha^n\rangle \tag{C-1}$$

where f_α are the steady-state time independent occupations obtained from the Floquet-Boltzmann equation and the instantaneous physical electronic energy is defined from the expectation value of the instantaneous Hamiltonian, $E(t) = \mathrm{Tr}[\hat{\rho}(t)\hat{h}_t]$.

For transparency, consider first magnetic field free single band driven Floquet problem [1] with the momentum state label $|\psi_\alpha^F(t)\rangle \equiv |\psi_{\mathbf{k}}^F(t)\rangle = \exp[-i\int_0^t dt' \epsilon(\mathbf{k} - e\mathbf{A}(t)/\hbar)]\,|\mathbf{k}\rangle$. Using the Schrodinger equation, we straightforwardly find:

$$\langle E(t)\rangle_T = \langle \sum_{\mathbf{k}} f_{\mathbf{k}} \langle\psi_{\mathbf{k}}^F(t)|\hat{h}(t)|\psi_{\mathbf{k}}^F(t)\rangle\rangle_T = \langle \sum_{\mathbf{k}} f_{\mathbf{k}} \langle\psi_{\mathbf{k}}^F(t)|i\hbar\partial_t|\psi_{\mathbf{k}}^F(t)\rangle\rangle_T = \sum_{\mathbf{k}} f_{\mathbf{k}}\epsilon_{\mathbf{k}}^F. \tag{C-2}$$

Thus, in this special zero-field case, the cycle-averaged physical energy equals the occupation-weighted Floquet energy.

For the finite magnetic field Landau level problem, substitution of the Floquet density matrix and employing Eq. (7) of the main text as

$$|\psi_\alpha^F(t)\rangle \equiv |N,r,t\rangle = e^{-iE_N(t-t_0)/\hbar-(i/\hbar)\int_{t_0}^t \Delta(t')\,dt'} D[\alpha(t)]\,|N\rangle \otimes |r\rangle\,, \tag{C-3}$$

we find the cycle averaged energy as

$$\begin{aligned}\langle E(t)\rangle_T &= \langle \sum_N f_N \langle\psi_N^F(t)|\hat{h}(t)|\psi_N^F(t)\rangle\rangle_T = \langle \sum_N f_N \langle\psi_N^F(t)|i\hbar\partial_t|\psi_N^F(t)\rangle\rangle_T \\ &= \sum_N f_N \left(E_N + i\hbar\left\langle \langle N|D^\dagger[\alpha(t)]e^{i\gamma(t)}\partial_t(e^{-i\gamma(t)}D[\alpha(t)])\,|N\rangle\right\rangle_T\right) \\ &= \sum_N f_N \left(E_N + i\hbar\left\langle -i\dot{\gamma}(t) + \langle N|D^\dagger[\alpha(t)]\dot{D}[\alpha(t)]|N\rangle\right\rangle_T\right)\,,\end{aligned} \tag{C-4}$$

where we used the Schrodinger equation in the first line and we decomposed $\Delta(t) = \bar{\Delta} + \delta\Delta(t)$ with $\langle\delta\Delta(t)\rangle_T = 0$ [see Eq. (B-10)] to introduce $\gamma(t) = (1/\hbar)\int_{t_0}^t \delta\Delta(t')\,dt'$. Notice cancellation of $\bar{\Delta}$ as well as $\langle\dot{\gamma}(t)\rangle_T = 0$. Using result derived in Appendix B displayed in Eq. (B-8) we find:

$$D^\dagger\dot{D} = \dot{\alpha}a^\dagger - \dot{\alpha}^* a + \frac{1}{2}\left(\alpha^*\dot{\alpha} - \alpha\dot{\alpha}^*\right), \qquad \langle N|D^\dagger\dot{D}|N\rangle = \frac{1}{2}\left(\alpha^*\dot{\alpha} - \alpha\dot{\alpha}^*\right) \tag{C-5}$$

where we used $\langle N|a|N\rangle = \langle N|a^\dagger|N\rangle = 0$. By explicitly employing $\alpha(t) = \alpha_+ e^{-i\Omega t} + \alpha_- e^{i\Omega t}$ [see Appendix B] we finally find

$$\langle E(t)\rangle_T = \sum_N f_N E_N + n_e\hbar\Omega(|\alpha_+|^2 - |\alpha_-|^2), \tag{C-6}$$

where $n_e = \sum_N f_N$. Notice, the second term only produces a smooth field-dependent offset in the cycle-averaged energy and does not affect the oscillatory structure discussed in the main text. This justifies using the effective energy $E_{\text{eff}} = \sum_N f_N E_N$ in Eq. (10).

2. Floquet entropy

The periodic Gibbs ensemble from Eq. (2) of the main text, that characterizes the non-equilibrium steady state, is a non-interacting mixed state of fermions. Namely, in second quantization it should be interpreted as saying that at a given time t, the time-dependent single-particle state associated with $\psi_\alpha^F(t)$ in Eq.(2), is occupied with a probability f_α and empty with probability $1 - f_\alpha$, where f_α is the time-independent function that solves the Boltzmann Eq. (3). Therefore, interesting, the Von-Neumann entropy of this periodic Gibbs ensemble is a time independent function given by the same elementary formula as for a free fermion gas. We review in this appendix its derivation. For each fermionic mode α, there are two possible occupation states, $|0_\alpha, t\rangle, |1_\alpha, t\rangle = c_\alpha^\dagger(t)|0_\alpha, t\rangle$. If the occupation of this mode is f_α, then the reduced density matrix of this mode is

$$\hat{\rho}_\alpha(t) = (1 - f_\alpha)\,|0_\alpha; t\rangle\,\langle 0_\alpha; t| + f_\alpha\,|1_\alpha; t\rangle\,\langle 1_\alpha; t|\,. \tag{C-7}$$

For a non-interacting diagonal ensemble, the full many-body density matrix factorizes over fermionic modes:

$$\hat{\rho}_{\text{MB}}(t) = \prod_\alpha \hat{\rho}_\alpha(t) = \prod_\alpha \left[(1 - f_\alpha)\,|0_\alpha; t\rangle\,\langle 0_\alpha; t| + f_\alpha\,|1_\alpha; t\rangle\,\langle 1_\alpha; t|\right]. \tag{C-8}$$

Equivalently, one can write the density matrix in the basis of many-body Slater determinants. A many-body configuration is specified by occupation numbers $\mathbf{n} = \{n_\alpha\}$, with $n_\alpha \in \{0, 1\}$ with the corresponding Slater determinant

$$|\mathbf{n}; t\rangle = \prod_\alpha \left[\hat{c}_\alpha^\dagger(t)\right]^{n_\alpha} |0\rangle\,. \tag{C-9}$$

Because the modes are independent, the probability of the Slater determinant $|\mathbf{n}, t\rangle$ is $P[\mathbf{n}] = \prod_\alpha f_\alpha^{n_\alpha}(1 - f_\alpha)^{1-n_\alpha}$. Thus the many-body density matrix can also be written as $\hat{\rho}_{\text{MB}}(t) = \sum_{\mathbf{n}} P[\mathbf{n}]|\mathbf{n}, t\rangle\langle\mathbf{n}, t|$. Which results in the von Neumann entropy of this many-body density matrix to become

$$S = -k_B\text{Tr}[\hat{\rho}_{\text{MB}}(t)\log\hat{\rho}_{\text{MB}}(t)] = -k_B\sum_{\mathbf{n}} P[\mathbf{n}]\log P[\mathbf{n}] = -k_B\sum_\alpha\left[f_\alpha\log f_\alpha + (1 - f_\alpha)\log(1 - f_\alpha)\right], \tag{C-10}$$

where we used $\sum_{\mathbf{n}} P[\mathbf{n}]n_\alpha = f_\alpha, \sum_{\mathbf{n}} P[\mathbf{n}] = 1$. Notice, this entropy is independent of time. The Floquet orbitals evolve periodically, but the eigenvalues of the many-body density matrix are the probabilities $P[\mathbf{n}]$, which are determined only by the time-independent occupations f_α.

For the driven Landau-level problem considered in the main text, the generic Floquet label is $\alpha = (N, r)$, where N is the Landau-level index and r is the guiding-center label introduced in Eq. (7). Magnetic translational symmetry makes the occupation independent of r, $f_{N,r} = f_N$. Therefore, for this case,

$$S = -k_B N_\phi \sum_N \left[f_N \log f_N + (1-f_N)\log(1-f_N)\right], \tag{C-11}$$

where $N_\phi = \sum_r 1$ is the number of fluxes piercing the 2D plane.

Appendix D: Scattering amplitudes in Landau levels

In this section, we provide a detailed procedure for performing the summation over the guiding center index. In the undriven Landau level problem, states are labeled as $|\alpha\rangle \equiv |N, r\rangle$, where N is the main Landau level cyclotron index, and r is the guiding center index. Due to magnetic translational invariance, we will show that scattering rates that appear in the kinetic equation can be effectively reduced to depend only on N, and the steady state occupation can be taken to only depends on N. We also use the equal coupling strength for all the bath modes in the system with $\hat{\chi}_q = \chi_0 |\mathbf{R}\rangle\langle\mathbf{R}|$, where $|\mathbf{R}\rangle$ is a fermion position eigenket and $\int_{\mathbf{R}} \equiv \int d^d\mathbf{R}/V$, which we introduced in the main text. Following Eq.(4) from the main text, our task is to evaluate the Floquet matrix element:

$$\int_{\mathbf{R}} \sum_{rr_1} \left| \sum_l \langle \phi^l_{N,r}| \chi_q |\phi^{l-p}_{N_1,r_1}\rangle \right|^2 = \chi_0^2 \int_{\mathbf{R}} \sum_{rr_1} \left| \sum_l \langle \phi^l_{N,r}|\mathbf{R}\rangle\langle\mathbf{R}|\phi^{l-p}_{N_1,r_1}\rangle \right|^2 . \tag{D-1}$$

We employ the identity $|\mathbf{R}\rangle\langle\mathbf{R}| = \delta(\mathbf{R}-\hat{\mathbf{R}}) = (1/A)\sum_q e^{iq(\mathbf{R}-\hat{\mathbf{R}})}$, where the position operator can be decomposed as $\hat{\mathbf{R}} = \hat{\mathbf{R}}_c + \hat{\mathbf{R}}_r$ into orbital, with $\hat{R}^x_c = l_B(\hat{a}+\hat{a}^\dagger)/\sqrt{2}, \hat{R}^y_c = -il_B(\hat{a}-\hat{a}^\dagger)/\sqrt{2}$, and guiding center index contributions with $\hat{R}^x_m = l_B(\hat{b}+\hat{b}^\dagger)/\sqrt{2}, \hat{R}^y_c = il_B(\hat{b}-\hat{b}^\dagger)/\sqrt{2}$ where $\hat{b}$ ($\hat{b}^\dagger$) are the guiding-center ladder operators. Thus we write:

$$\sum_{rr_1} \left| \sum_l \langle \phi^l_{N,r}| \chi_q |\phi^{l-p}_{N_1,r_1}\rangle \right|^2 = \chi_0^2 \sum_{rr_1} \sum_{ll'} \langle \phi^l_{N,r}|\mathbf{R}\rangle\langle\mathbf{R}|\phi^{l-p}_{N_1,r_1}\rangle \langle \phi^{l'-p}_{N_1,r_1}|\mathbf{R}\rangle\langle\mathbf{R}|\phi^{l'}_{N,r}\rangle \tag{D-2}$$

$$= \chi_0^2 \sum_{ll'} \sum_r \langle \phi^l_{N,r}|\mathbf{R}\rangle\langle\mathbf{R}|\phi^{l'}_{N,r}\rangle \sum_{r_1} \langle\mathbf{R}|\phi^{l-p}_{N_1,r_1}\rangle \langle \phi^{l'-p}_{N_1,r_1}|\mathbf{R}\rangle, \tag{D-3}$$

which can be further simplified by employing:

$$\sum_r \langle \phi^l_{N,r}|\mathbf{R}\rangle\langle\mathbf{R}|\phi^{l'}_{N,r}\rangle = \frac{1}{A}\sum_q \langle \phi^l_N| e^{iq(\mathbf{R}-\hat{\mathbf{R}}_c[a,a^\dagger])} |\phi^{l'}_N\rangle \sum_r \langle r| e^{-iq\hat{\mathbf{R}}_r[b,b^\dagger]} |r\rangle = \frac{N_\phi}{A} \langle \phi^l_N|\phi^{l'}_N\rangle \tag{D-4}$$

where we used $|\phi^{l'}_{N,r}\rangle = |\phi^{l'}_N\rangle \otimes |r\rangle$ and the fact that $\mathrm{Tr}[e^{iq\hat{\mathbf{R}}_r}] = N_\phi \delta_{q,0}$. By substitution of Eq.(D-4) into Eq.(4) we obtained Eq.(8) of the main text.

Appendix E: Evaluation of the trace in Eq.(9) of the main text and perturbative limit

Here we show the details of Eq.(9) derivation in the main text. First, we write by the definition

$$\sum_{c'} |\phi^{c'}_N\rangle\langle\phi^{c'-d}_{N_1}| = \int_0^T \int_0^T \frac{dtdt'}{T^2} \left[\sum_{c'} e^{ic'\Omega(t-t')}\right] e^{id\Omega t'} |N, \alpha(t)\rangle\langle N_1, \alpha(t')| \tag{E-1}$$

where $|N, \alpha(t)\rangle \equiv D[\alpha(t)]|N\rangle$and we recognize the Poisson summation formula:

$$\sum_{c'=-\infty}^{\infty} e^{ic'\Omega(t-t')} = \frac{2\pi}{\Omega}\sum_{n=-\infty}^{\infty} \delta\Big(t-t'-\frac{2\pi n}{\Omega}\Big) = T\sum_{n=-\infty}^{\infty}\delta\Big(t-t'-nT\Big) = T\delta\Big(t-t'\Big) \tag{E-2}$$

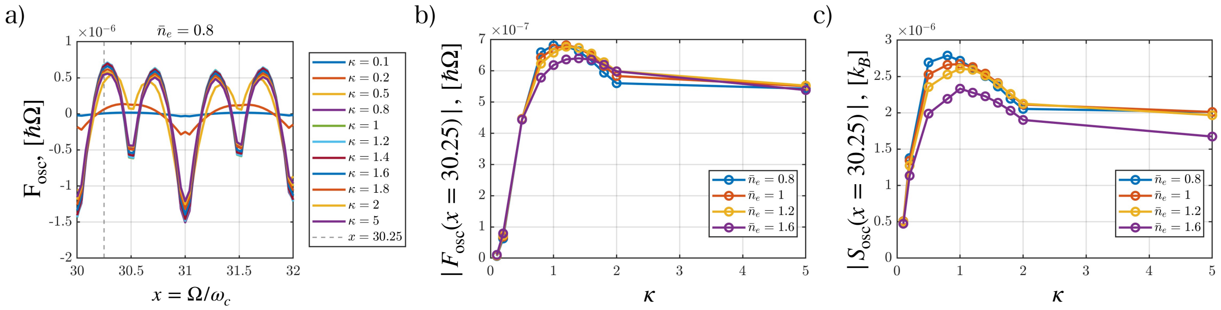

FIG. D-4. (a) Non-perturbative effective free-energy (see Eq. (10) of the main text) oscillations at $k_B T_{\mathrm{bath}} = 0.2\hbar\Omega$, computed with Floquet-harmonic cutoff $p_{\max} = 2$ in the non-perturbative Floquet-Boltzmann equation, Eq. (E-6). Amplitudes of the free-energy (b) and entropy (c) oscillations at $\Omega = 30.25\omega_c$ as a function of the drive strength κ, for the effective fillings $\bar{n}_e$ indicated in the legend. The Floquet oscillations saturate for strong driving, around $\kappa \sim 1$.

note, we used $-T < t - t' < T$ since $t \in [0, T)$, simplifying to

$$\sum_{c'} |\phi_N^{c'}\rangle \langle \phi_{N_1}^{c'-d}| = \int_0^T \frac{dt}{T} e^{id\Omega t} |N, \alpha(t)\rangle \langle N_1, \alpha(t)| \,. \tag{E-3}$$

Next, we employ the orthogonality condition [19]:

$$\langle N, \alpha(t')|N, \alpha(t)\rangle = \exp\left\{\frac{1}{2}\Big[\alpha(t')^*\alpha(t) - \alpha(t')\alpha(t)^*\Big] - \frac{1}{2}|\alpha(t) - \alpha(t')|^2\right\} L_N\Big(|\alpha(t) - \alpha(t')|^2\Big). \tag{E-4}$$

and straightforwardly find Eq.(9) of the main text where

$$\beta(x, y) = \frac{1}{\hbar}\left[\frac{z_+}{\omega_c - \Omega}(e^{-i2\pi x} - e^{-i2\pi y}) + \frac{z_-}{\omega_c + \Omega}(e^{i2\pi x} - e^{i2\pi y})\right]. \tag{E-5}$$

Finally, for the case of the circularly polarized light z_+ or z_- vanishes (RHS/LHS light polarization), and we write the simplified Floquet Boltzmann equation as

$$0 = \sum_{N=0}^{\infty}\left\{(1 - f_{N_1}) f_N \sum_{p=-\infty}^{\infty} w_{N,N_1}^p S[\epsilon_N^F - \epsilon_{N_1}^F + p\Omega] - (N_1 \leftrightarrow N)\right\}, \tag{E-6}$$

$$S[X] = \nu_B[X](N[X] + 1)\Theta[X] + \nu_B[-X]N[-X]\Theta[-X], \tag{E-7}$$

$$w_{N,N_1}^d = \int_0^1 du \cos[2\pi du] e^{-\gamma(u)} L_{N_1}\Big[\gamma(u)\Big] L_N\Big[\gamma(u)\Big], \qquad \gamma(u) = 2R_+(1 - \cos[2\pi u]) \tag{E-8}$$

note $w_{N,N_1}^{-d} = w_{N,N_1}^d$ and

$$R_+ = \frac{|z_+|^2}{(\omega_c - \Omega)^2} + \frac{|z_-|^2}{(\omega_c + \Omega)^2} \approx \frac{|z_+|^2}{\hbar^2\Omega^2} + \frac{|z_-|^2}{\hbar^2\Omega^2} \tag{E-9}$$

where $\kappa^2 = R_+\Omega/\omega_c$ is magnetic field independent, positively defined strength of the periodic drive. To see this explicitly employ:

$$\|z_+\|^2 = \frac{\hbar\omega_c e^2}{2m\Omega^2}(\|E_\Omega\|^2 + i[\mathbf{E}_\Omega \times \mathbf{E}_{-\Omega}]), \qquad \|z_-\|^2 = \frac{\hbar\omega_c e^2}{2m\Omega^2}(\|E_\Omega\|^2 - i[\mathbf{E}_\Omega \times \mathbf{E}_{-\Omega}]), \tag{E-10}$$

that follow from Eqs.(B-2-B-4).

Critically, in non-perturbative fashion we find saturation of the Floquet oscillations amplitude with amplitude of the drive at $\kappa \sim 1$. The non-perturbative regive is extremely numerically heavy and is out of the scope of current study.

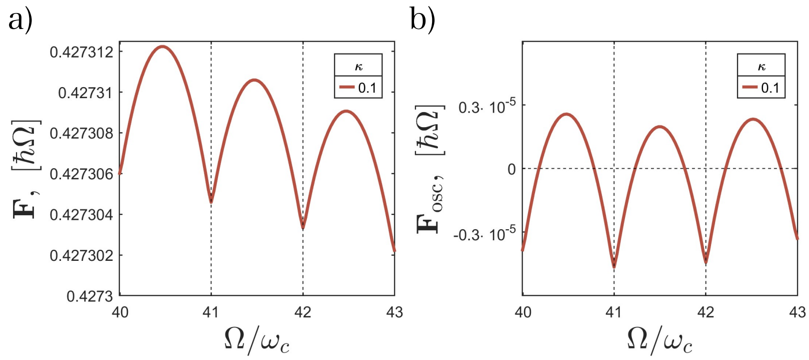

FIG. E-5. (a) Effective free energy, defined in Eq. (10) of the main text, as a function of magnetic field, parametrized by ω_c. (b) Corresponding isolated oscillatory component.

From Eq.(E-8) we obtain the scattering rate perturbative in the strength of the drive κ which is consistent with our previous study [2]:

$$w^d_{N,N_1} = \begin{cases} 1 - 2\,(1 + N_1 + N)\left(|\alpha_+|^2 + |\alpha_-|^2\right) + O(\mathcal{E}^4), & d = 0, \\ (1 + N_1 + N)\left(|\alpha_+|^2 + |\alpha_-|^2\right) + O(\mathcal{E}^4), & d = \pm 1, \\ 0 + O(\mathcal{E}^4), & |d| \geq 2. \end{cases} \tag{E-11}$$

Appendix F: Oscillation extraction

In this section we discuss the numerical calculation of the Floquet free energy oscillations introduced in Eq.(10) of the main text. We begin by fixing of the total particle density for each magnetic field. In 2D electron gas at $T = 0$, the electron density is related to the Fermi energy as $\mu = \pi\hbar^2 n_e/m$, while the Landau level degeneracy per area is $N_\phi/A = eB/h = eB/(2\pi\hbar)$ with the cyclotron frequency $\omega_c = eB/m$. Thus we impose the normalization:

$$\lim_{B\to 0} \frac{N_\phi}{A} \sum_N f(N) = n_e = \frac{m\mu}{\pi\hbar^2} = \left[\frac{N_\phi}{A}\right] \frac{m\mu A}{\pi\hbar^2 N_\phi} = \left[\frac{N_\phi}{A}\right] \frac{2m\mu}{\hbar e B} = \left[\frac{N_\phi}{A}\right] \frac{2\mu}{\hbar\omega_c}. \tag{F-1}$$

The quantity for which we compute the oscillations is the following effective free energy:

$$F = E - T_{\mathrm{bath}} S = E - T_{\mathrm{bath}} N_\phi S, \qquad \frac{F}{n_e A\hbar\Omega} = \frac{\omega_c^2}{\Omega^2 \bar{\mu}} \sum_N \left(f(\epsilon_N)(N + 1/2) - \frac{T}{\omega_c} S_N \right). \tag{F-2}$$

To extract the strictly oscillatory part, we further subtract the non-oscillatory (Landau diamagnetic) component to identify the quantum oscillations see Fig.E-5.

Appendix G: ohmic Bath Quantum oscillations

Here we show that for a perfectly ohmic bath (i.e c_2=0), the "high temperature oscillations" are absent. This is consistent with the results of Ref[3], which found that the second derivative discontinuity for the perfectly Ohmic bath is only sharp when the bath is at zero temperature and it is smeared out when the bath is at finite temperature. In agreement with this, we find that the quantum oscillations only are present in the low temperature regime in this case, as shown in Fig.G-6.

Appendix H: Oscillations Floquet regime phase shift

We contrast the oscillation phases in the deep Floquet regime to the equilibrium quantum oscillations. To do this we choose the electron density and the driving frequency to have a value such that the area of the usual equilibrium

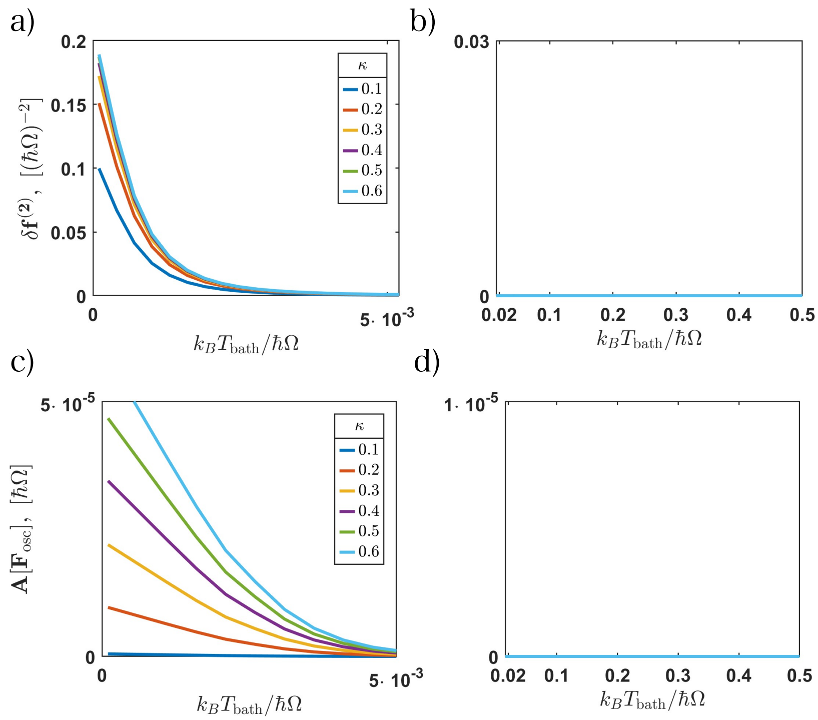

FIG. G-6. Jump of the second derivative of the occupation function at $k = k_F$ for a perfectly ohmic bath, $c_2 = 0$, as a function of bath temperature and for several drive strengths κ in (a,b). Corresponding quantum-oscillation amplitude in the interval $\Omega/\omega_c \in [40, 43]$ in (c,d). We use $\bar{n}_e = 0.8$. In contrast to Fig. 2 of the main text, the high-temperature revival is absent for the perfectly ohmic bath.

Fermi surface and the Floquet Fermi surface are identical, and thus their quantum oscillations frequency is the same. The idea is that this allows us to contrast more easily their phase to determine if there are any changes. We find, however, that the phase as a function of magnetic field is preserved during the crossover from usual oscillations which occurs as the driving amplitude, κ, increases, as shown in FIG.I-7, which illustrates that minima and maxima do not shift as the amplitude of the drive, κ increases. Therefore, the phase of these of oscillations is not consistent with the one observed in the regime of coexistence of MIRO and SdH oscillations reported in Ref.[27].

Appendix I: Fit parameters for the low and hight temperature discontinuity behavior

Here we present values of the parameter fit to for the low and hight temperature occupation function discontinuity fits, see Table I-I.

	Low temperature fit			**Higher temperature fit**	
κ	a	b	c	a'	b'
0.1	0.097	0.3	4.4×10^{-4}	9×10^{-4}	8×10^{-3}
0.3	0.15	-0.7	4.4×10^{-4}	6×10^{-3}	3.6×10^{-2}
0.4	0.165	1.06	4.4×10^{-4}	1.6×10^{-2}	0.1

TABLE I-I. Fitted parameters for the jump in the second derivative of the occupation function. In the low-temperature regime, region I, the jump is fitted by $\delta f^{(2)}[\tau \to 0] = (a + b\tau)e^{-\tau/c}$. In the high-temperature regime, region II, it is fitted by $\delta f^{(2)}[\tau \gg \tau_{\max}] = a'\tau/(b' + \tau^2)$. For a comparison between the fits and the numerically simulated data, see Fig. 3c.

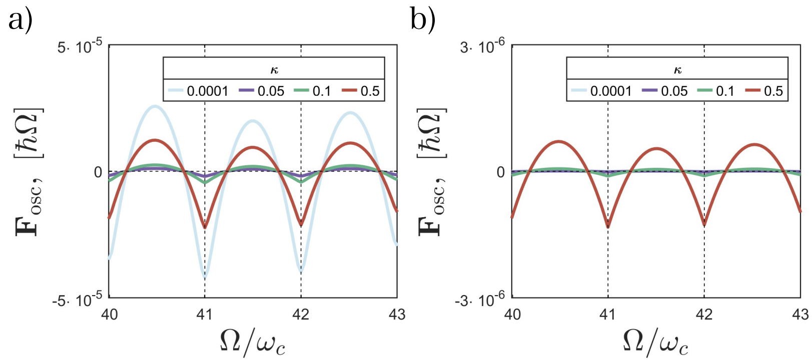

FIG. I-7. (a) Low-temperature quantum oscillations at $k_B T_{\mathrm{bath}} = 10^{-4}\hbar\Omega$. We find that the oscillation phase remains unchanged across the crossover from the conventional equilibrium regime (blue line) to the Floquet regime (red line). (b) High-temperature regime at $k_B T_{\mathrm{bath}} = 0.1\hbar\Omega$. The equilibrium oscillations are exponentially suppressed and remain close to the $F_{\mathrm{osc}} = 0$ line, while the Floquet-controlled component grows with increasing drive strength. Here we choose $\bar{n}_e = 1$, so that the equilibrium Fermi energy is equal to $\hbar\Omega$.