SUPPLEMENTAL MATERIAL FOR "THE FLOQUET FERMI LIQUID"

Appendix A: Density matrix of a general system Hamiltonian coupled to the featureless fermionic bath

In Refs. [13, 32], we focused on analyzing the case of a diagonal system Hamiltonian. In the current study, we broaden our scope by considering more general, non-diagonal system Hamiltonian.

Following Eqs. (2) to (16) from Ref. [13], we have the open-system Schrödinger equation for a generic system coupled to the featureless fermionic bath:

$$i\partial_t |\psi_n^{(j)}(t)\rangle = \left[H_S(t) - i\Gamma\right] |\psi_n^{(j)}(t)\rangle + \lambda e^{-i\varepsilon_j(t-t_0)} |\chi_n\rangle\,, \tag{A-1}$$

where λ is the tunneling amplitude onto the bath, and $\Gamma = \nu_0\lambda^2/2$, with ν_0 the density of states of the bath. This equation is a non-Hermitian version of the Schrödinger equation in which the system Hamiltonian is dressed by a constant imaginary part "$-i\Gamma$", accounting for the decay into the bath. And it crucially includes an inhomogeneous term (the one proportional to λ) which accounts for the bath feedback effect (see Refs. [13, 32] for details). The one-body density matrix projected onto the system at time t is then given by

$$\rho_S(t) = \sum_{n,j} f_0(\varepsilon_j) |\psi_n^{(j)}(t)\rangle \langle\psi_n^{(j)}(t)|\,, \tag{A-2}$$

Let's denote the time evolution operator of the closed system by $U_S(t,t')$, so that it satisfies the following equation:

$$i\partial_t U_S(t,t') = H_S(t)U_S(t,t'). \tag{A-3}$$

Next, we define an auxiliary state $|\tilde{\psi}_n^{(j)}(t)\rangle$ from the state $|\psi_n^{(j)}(t)\rangle$ as follows:

$$|\psi_n^{(j)}(t)\rangle \equiv e^{-\Gamma(t-t_0)}U_S(t,t') |\tilde{\psi}_n^{(j)}(t)\rangle\,. \tag{A-4}$$

By substituting Eq. (A-4) into Eq. (A-1), we derive the open-system Schrödinger equation for the auxiliary state $|\tilde{\psi}_n^{(j)}(t)\rangle$:

$$i\frac{\partial}{\partial t} |\tilde{\psi}_n^{(j)}(t)\rangle = \lambda U_S(t',t) |\chi_n\rangle\, e^{-i(\varepsilon_j+i\Gamma)(t-t_0)}, \tag{A-5}$$

where we used the identity $U_S(t,t')U_S(t',t) = U_S(t,t) = 1$. The solution for $|\tilde{\psi}_n^{(j)}(t)\rangle$ is therefore:

$$|\tilde{\psi}_n^{(j)}(t)\rangle = -i\lambda \int_{t_0}^{t} U_S(t',t'') |\chi_n\rangle\, e^{-i(\varepsilon_j+i\Gamma)(t''-t_0)} dt''. \tag{A-6}$$

Thus, the solution for the amplitude of the open system problem is

$$|\psi_n^{(j)}(t)\rangle = -i\lambda e^{-\Gamma(t-t_0)} \int_{t_0}^{t} U_S(t,t') |\chi_n\rangle\, e^{-i(\varepsilon_j+i\Gamma)(t'-t_0)} dt' \tag{A-7}$$

By inserting Eq. (A-7) into Eq. (A-2) and taking $t_0 \to -\infty$ to obtain the late time steady state, we achieve the Eq. (2) in the main text:

$$\rho_S(t) = \Gamma \int_{-\infty}^{+\infty} \frac{\mathrm{d}\epsilon}{\pi} f_0(\epsilon) U_\Gamma(t,\epsilon) U_\Gamma^\dagger(t,\epsilon), \quad U_\Gamma(t,\epsilon) = \int_{-\infty}^{t} \mathrm{d}t' e^{\Gamma(t'-t)-i\epsilon t'} U_S(t,t'), \tag{A-8}$$

where we utilized the property that the featureless fermionic bath has a constant density of state $\nu_B(\epsilon) = 2\pi\sum_j \delta(\epsilon - \varepsilon_j) \equiv \nu_0$, and used $\Gamma \equiv \lambda^2\nu_0/2$.

We note that we have used an initial condition in writing Eq. (A-6) so that $|\psi_n^{(j)}(t_0)\rangle = 0$, namely that in a distant past all particles are located in the bath. This assumption is convenient because it eliminates the transient part of the solutions, but it is not strictly necessary. This is because Eq. (A-1) is a inhomogeneous equation for which one can always add any solution to the homogeneous equation (the one with $\lambda = 0$) in order to satisfy any given initial condition. But such solutions of the homogeneous equation would all decay to zero as $t \to \infty$ for any finite $\Gamma > 0$. This can be viewed as a type of "irreversible radiation" of the information of the initial state of the system onto the bath, which eventually erases all such information leading to the unique late-time steady state from Eq. (A-8).

Appendix B: Density matrix of a Floquet system Hamiltonian coupled to the ideal fermionic bath

We will now restrict to periodic Hamiltonians, $H(t) = H(t+T)$. We will consider the situation when the coupling to the bath is weak (as indicated by $\Gamma \to 0$), such that the bath operates as an "ideal" thermal bath. Our objective is to demonstrate that under these conditions, Eq. (2) from the main text simplifies to Eq. (3).

First, we denote Floquet eigenstates as $|\psi_a^F(t)\rangle$ satisfying:

$$i\partial_t |\psi_a^F(t)\rangle = H_S(t) |\psi_a^F(t)\rangle . \tag{B-1}$$

From the Floquet's theorem, $|\psi_a^F(t)\rangle$ can be expressed in terms of its Floquet harmonics:

$$|\psi_a^F(t)\rangle = \sum_l e^{-i(\epsilon_a^F + l\Omega)t} |\varphi_{a,l}\rangle , \quad \Omega = 2\pi/T. \tag{B-2}$$

Using this formulation, we can express the unitary evolution operator for the system, denoted as $U_S(t,t')$, as follows:

$$U_S(t,t') = \sum_a |\psi_a^F(t)\rangle \langle \psi_a^F(t')| = \sum_{a,l_1,l_2} e^{-i\epsilon_a^F(t-t')} e^{-i\Omega(l_1 t - l_2 t')} |\varphi_{a,l_1}\rangle \langle \varphi_{a,l_2}| . \tag{B-3}$$

Next, we substitute Eq. (B-3) into Eq. (A-8) and obtain:

$$U_\Gamma(t,\epsilon) = \sum_{a,l_1,l_2} |\varphi_{a,l_1}\rangle \langle \varphi_{a,l_2}| \frac{e^{-i\Omega(l_1-l_2)t} e^{-i\epsilon t}}{i\epsilon_a^F + i\Omega l_2 - i\epsilon + \Gamma}. \tag{B-4}$$

Upon integrating over ϵ, we obtain:

$$\begin{aligned} \rho_S(t) = \sum_{a,b,l_1,l_2} |\varphi_{a,l_1}\rangle e^{-i\Omega l_1 t} \Bigg(\sum_{l_3,l_4} \langle \varphi_{a,l_3} | \varphi_{b,l_4}\rangle e^{-i\Omega(l_4-l_3)t} \\ \times \frac{\Gamma}{2\Gamma - i(l_4\Omega + \epsilon_b^F - l_3\Omega - \epsilon_a^F)} [f_+(\epsilon_a^F + l_3\Omega) + f_-(\epsilon_b^F + l_4\Omega)] \Bigg) e^{i\Omega l_2 t} \langle \varphi_{b,l_2}| , \end{aligned} \tag{B-5}$$

where $f_+(\epsilon) = [f_-(\epsilon)]^*$ and they are given by:

$$f_\pm(\epsilon) = \frac{1}{2} \pm \frac{i}{\pi} \Psi^{(0)}\Big(\frac{1}{2} \pm i\beta \frac{\epsilon \mp i\Gamma - \mu}{2\pi}\Big), \tag{B-6}$$

with $\Psi^{(0)}$ the 0-th order Polygamma function (or the digamma function).

Assuming a weak coupling to the bath (i.e., $\Gamma \to 0$), only the terms that satisfy the equation $l_4\Omega + \epsilon_b^F - l_3\Omega - \epsilon_a^F = 0$ are retained. Provided that Floquet band crossings are avoided, the equality $l_4\Omega + \epsilon_b^F - l_3\Omega - \epsilon_a^F = 0$ imposes the conditions $a = b$ and $l_3 = l_4$. Consequently, the density matrix simplifies to:

$$\lim_{\Gamma\to 0} \rho_S(t) = \sum_a p_a |\psi_a^F(t)\rangle \langle \psi_a^F(t)| , \quad p_a = \sum_l \langle \varphi_{a,l} | \varphi_{a,l}\rangle f_0(\epsilon_a^F + l\Omega). \tag{B-7}$$

where we used the limit

$$\lim_{\Gamma\to 0} \frac{1}{2}\Big[f_+(\epsilon) + f_-(\epsilon)\Big] = f_0(\epsilon). \tag{B-8}$$

in obtaining Eq. (B-7). From the above we can see that Eq. (2) indeed simplifies to Eq. (3) in the main text.

Appendix C: Floquet perturbation theory

1. General theory

In this Section, we introduce the detailed scheme for how to perturbatively solve the Schrödinger equation in the presence of a periodic drive, where the state's dynamics is defined by (for instructive purposes in this part of the appendix, we restore the units):

$$i\hbar\partial_t |\psi_a(t)\rangle = H_S(t) |\psi_a(t)\rangle = (H_0 + V(t)) |\psi_a(t)\rangle . \tag{C-1}$$

Due to the time periodicity of the Hamiltonian $H_S(t) = H_S(t+T)$ we can use the following Floquet expansion for the states:

$$|\psi_a(t)\rangle = e^{-i\epsilon_a^F(t-t_0)/\hbar} \sum_{l'=-\infty}^{\infty} e^{-il'\Omega(t-t_0)} \sum_b c_{a,b}^{l'} |\chi_b\rangle \tag{C-2}$$

where ϵ_a^F is the Floquet energy of the state and $|\chi_b\rangle$ are the solutions of the unperturbed problem $H_0|\chi_b\rangle = E_b^{(0)}|\chi_b\rangle$. Essentially, we expanded the r.h.s. of the Eq.(B-2) in the known basis $|\varphi_{a,l}\rangle = \sum_b c_{a,b}^l |\chi_b\rangle$. Since the Floquet energy is not uniquely defined but up to a shift on $k\hbar\Omega$ where $k \in \mathbb{Z}$ [$k\Omega$ can be absorbed into the dummy index l' in Eq.(C-2)], without losing the generality, we fix this ambiguity or the Floquet gauge by setting $c_{a,b}^{l,0} = \delta_{l,0}\delta_{a,b}$ in the absence of the perturbation. Meaning that state $|\psi_a(t)\rangle$ is adiabatically connected to the unperturbed state $|\chi_a(t)\rangle$.

Next, we substitute Eq.(C-2) into Eq.(C-1) and project the equation onto mode l obtaining:

$$(E_a^{(0)} + l\hbar\Omega - H_0) \sum_b c_{a,b}^l |\chi_b\rangle = \int_0^T \frac{dt}{T} \sum_{l'=-\infty}^{\infty} (V(t) - \Delta_a) e^{i(l-l')\Omega(t-t_0)} \sum_b c_{a,b}^{l'} |\chi_b\rangle . \tag{C-3}$$

where $\Delta_a = \epsilon_a^F - E_a^{(0)} = \mathcal{O}(V^1)$ is a perturbation induced correction to the a-th state's energy. The form of the equation above is convenient since right hand side $V(t) - \Delta_a$ is at least of the order of the perturbation, while $E_a^{(0)} + l\hbar\Omega - H_0$ is non-perturbative. Eq.(C-3) is sufficient to determine all the Floquet amplitudes and energies.

By projecting Eq.(C-3) on the state $\langle\chi_a|$:

$$(\Delta_a + l\hbar\Omega) c_{a,a}^n = \langle\chi_a| \int_0^T \frac{dt}{T} \sum_{l'=-\infty}^{\infty} V(t) e^{i(l-l')\Omega(t-t_0)} \sum_b c_{a,b}^{l'} |\chi_b\rangle , \tag{C-4}$$

we obtain the equation used to determine the Δ_a. For $l \neq 0$ from projection of the Eq.(C-3) onto a state different from a, namely $\langle\chi_c| \neq \langle\chi_a|$ we obtain:

$$c_{a,c}^l = \frac{1}{E_a^{(0)} - E_c^{(0)} + l\hbar\Omega} \langle\chi_c| \int_0^T \frac{dt}{T} \sum_{l'=-\infty}^{\infty} (V(t) - \Delta_a) e^{i(l-l')\Omega(t-t_0)} \sum_b c_{a,b}^{l'} |\chi_b\rangle , \tag{C-5}$$

which is correct for any a, c assuming that $E_c^{(0)} - E_a^{(0)} \neq l\hbar\Omega$ for any a, b and l. The above is used to obtain coefficients $c_{a,c}^{l\neq 0}$. For $l = 0$, the inverse operator $H - E_a^{(0)}$ is well defined on the space orthogonal to $|\chi_a\rangle$. Thus, using $P_a^\perp = \sum_{d\neq a} |\chi_d\rangle\langle\chi_d|$, we can write:

$$c_{a,c}^0 = \delta_{a,c} + \langle\chi_c| \frac{P_a^\perp}{E_a^{(0)} - H_0} \int_0^T \frac{dt}{T} \sum_{l'=-\infty}^{\infty} (V(t) - \Delta_N) e^{-il'\Omega(t-t_0)} \sum_b c_{a,b}^{l'} |\chi_b\rangle , \tag{C-6}$$

that determines remaining $c_{a,c}^{l=0}$ coefficients. Note, $\delta_{a,c}$ was added as a solution of the homogeneous equation.

By using the perturbation expansions of states and energies in powers of their smallness:

$$\begin{aligned} c_{a,c}^l &= \delta_{l,0}\delta_{a,c} + c_{a,c}^{l,(1)} + c_{a,c}^{l,(2)} + \ldots, \\ \Delta_a &= \Delta_a^{(1)} + \Delta_a^{(2)} + \ldots, \end{aligned} \tag{C-7}$$

where the superscript (n) indicates the correction's order, we find perturbed states $|\psi_a(t)\rangle$. After these states are normalised, we find the Floquet amplitudes using their definition in Eq.(3) of the main text.

2. Application to parabolic fermions

We now apply our theory to parabolic fermions coupled simultaneously to a constant magnetic and periodic electric field. The corresponding Hamiltonian is $H(\mathbf{k}, t) = [\hbar\mathbf{k} - e\mathbf{A}_0(\mathbf{r}) - e\mathbf{A}(t)]^2/(2m)$, which using

$$\boldsymbol{\pi} = \hbar\mathbf{k} - e\mathbf{A}_0(\mathbf{r}), \qquad a = \frac{l_c}{\sqrt{2}\hbar}(\pi_x - i\pi_y), \qquad a^\dagger = \frac{l_c}{\sqrt{2}\hbar}(\pi_x + i\pi_y), \qquad [a, a^\dagger] = 1, \qquad l_c = \sqrt{\frac{c\hbar}{eB}}, \tag{C-8}$$

we rewrite as:

$$H(\mathbf{k},t) = \hbar\omega_c a^\dagger a - a z_t^* - a^\dagger z_t + c(t), \tag{C-9}$$

here $z_t = e\sqrt{\hbar\omega_c/(2m)}(A^x(t) - iA^y(t))$, $c(t) = \hbar\omega_c/2 + e^2\mathbf{A}(t)^2/2m$, ω_c is the cyclotron frequency, m is the electron mass. The solutions of the above equation, in the absence of the electric field, are the Landau level states.

We consider $V(t)$ to be small. Note, the magnetic field is treated non-perturbatively and is part of $H_0 = \hbar\omega_c a^\dagger a$. We absorb the time-dependant constant term of the Hamiltonian, $c(t)$, into a phase of the wave-function as follows $|\psi_N(t)\rangle = e^{-iC(t)}|N(t)\rangle$, where $C(t) = c_0 + \int_{t_0}^t c(t')dt'/\hbar$. Now, we can write:

$$i\hbar\frac{d}{dt}|N(t)\rangle = (H_0 + V(t))|N(t)\rangle, \qquad V(t) = -a(z_+^* e^{i\Omega t} + z_-^* e^{-i\Omega t}) - a^\dagger(z_+ e^{-i\Omega t} + z_- e^{i\Omega t}). \tag{C-10}$$

Next we apply the theory from Eq. (C-1-C-6) for $|\chi_b\rangle = |M\rangle^{(0)}$ to be unperturbed Landau states with $E_M^{(0)} = M\hbar\omega_c, M \in [0\ldots+\infty]$, where $\hbar\omega_c/2$ energy shift was absorbed to the phase $C(t)$. We find the first order coefficients to be:

$$\begin{aligned} c_{N,N-1}^{1,(1)} &= -\frac{z_-^*\sqrt{N}}{\hbar\omega_c + \hbar\Omega}, \quad c_{N,N+1}^{1,(1)} = -\frac{z_+\sqrt{N+1}}{-\hbar\omega_c + \hbar\Omega}, \\ c_{N,N-1}^{-1,(1)} &= -\frac{z_+^*\sqrt{N}}{\hbar\omega_c - \hbar\Omega}, \quad c_{N,N+1}^{-1,(1)} = -\frac{z_-\sqrt{N+1}}{-\hbar\omega_c - \hbar\Omega}, \end{aligned} \tag{C-11}$$

and $\Delta_N^{(1)} = 0$. Note the states are yet to be normalised. All the coefficients beyond those appearing above are second or higher order in powers of electric field, which contribute as at least of third order correction to the amplitudes. The second order correction to the energy is found to be $\Delta_N^{(2)} = -|z_+|^2/(\hbar\omega_c - \hbar\Omega) - |z_-|^2/(\hbar\omega_c + \hbar\Omega)$, where:

$$|z_\pm|^2 = \frac{\hbar\omega_c e^2}{2m\Omega^2}(|\mathbf{E}|^2 \pm i[\mathbf{E}\times\mathbf{E}^*]_z). \tag{C-12}$$

After the state normalization using Eqs. (C-11) we find the the Floquet amplitudes up to the second order in powers of the electric field, which are given by:

$$|\varphi_{N,\pm1}|^2 = \frac{\epsilon_N^F}{\hbar\omega_c}R_+ \pm \frac{R_-}{2} + \mathcal{O}(E^4), \qquad |\varphi_{N,0}|^2 = 1 - 2\frac{\epsilon_N^F}{\hbar\omega_c}R_+ + \mathcal{O}(E^4), \tag{C-13}$$

and the Floquet energy of the N-th Landau level given by $\epsilon_N^F = (N+1/2)\hbar\omega_c + \Delta E$, where

$$R_\pm = \frac{|z_+|^2}{(\hbar\omega_c - \hbar\Omega)^2} \pm \frac{|z_-|^2}{(\hbar\omega_c + \hbar\Omega)^2}, \qquad \Delta E = \frac{e^2|\mathbf{E}|^2}{m\Omega^2} - \frac{|z_+|^2}{\hbar\omega_c - \hbar\Omega} - \frac{|z_-|^2}{\hbar\omega_c + \hbar\Omega}, \tag{C-14}$$

We can see that the oscillating electric field produces a uniform (N independent) energy shift to all the Landau levels energies, which effectively redefines the chemical potential. Note, the dominant contribution to the oscillation of the G_{eff} comes from the terms in summation over N (see Eq.(6)), when $\epsilon_N^F \approx \mu$, namely levels close to the Fermi Surface. For large chemical potentials, $\epsilon_N^F/\hbar\omega_c \gg 1$, the factor R_- in Eq.(C-13) is negligible, yet it is interesting to note that R_- is responsible for the imbalance of the occupation of the $l=1$ vs $l=-1$ Floquet Fermi surfaces. Notice also that both $R_\pm$, according to Eq.(C-12), are sensitive to the electric field polarization and, therefore, can be controlled by changing the degree of the polarization (e.g. linear vs circular) of the driving electric field.

Appendix D: Magnetic oscillations

In this Section, we show the detailed derivation of the system magnetic oscillations of several quantities, including the time-averaged system energy. We will employ the following Poisson summation formula:

$$\sum_{N=0}^{\infty} F(N+1/2) = \int_0^\infty F(x)dx + 2\sum_{k=1}^{\infty}\int_0^\infty (-1)^k F(x)\cos(2\pi kx)dx. \tag{D-1}$$

Let us assume that the function $F(x) = f_0(x)g(x)$ can be represented as a multiplication of the f_0, Fermi-Dirac function, with a real function $g(x)$. Then the second integral of the r.h.s. of the Eq.(D-1) can be rewritten as:

$$\begin{aligned}\int_0^\infty F(x)\cos(2\pi kx)dx &= \int_0^\infty f_0(x)g(x)\cos(2\pi kx)dx = \int_0^\infty f_0(x)d\Big[\int_0^x g(y)\cos(2\pi ky)\Big] \\ &= -\int_0^\infty \frac{df_0(x)}{dx}\Big[\int_0^x g(y)\cos(2\pi ky)dy\Big]dx \approx -\text{Re}\Big\{\int_{-\infty}^\infty \frac{df_0(x)}{dx}\Big[\int_0^x g(y)e^{2\pi iky}dy\Big]dx\Big\},\end{aligned} \tag{D-2}$$

where the surface term vanishes since the $f_0(\infty) \to 0$ and term at $x = 0$ is zero due to the integral in the brakets. We replaced the lower limit of the integration to $-\infty$, which is a good approximation if $\mu/\hbar\omega_c \gg 1$. In the last step, we also rewrote $\cos(2\pi kx)$ as an exponent since this form simplifies the integration over x. We aim to compute the magnetic oscillations up to the second order in powers of the electric field, also assuming $\mu \gg \hbar\omega_c$, $\mu \gg k_B T_0 = \beta^{-1}$.

Interestingly, the expression in the square brackets in the last part of Eq.(D-2) is the zero-temperature result of the full expression if we set $x = \mu$ (note for $T_0 = 0$ $df_0[\epsilon]/d\epsilon \sim \delta(\epsilon - \mu)$). This is the reason why calculations for the magnetic oscillations are often carried out at zero temperature and, later on, weighted with the spectral weight of the Fermi-Dirac function:

$$\frac{\partial f_0}{\partial \epsilon}[\epsilon] = -\int_{-\infty}^\infty \frac{d\lambda}{2\pi}\frac{\pi\lambda/\beta}{\sinh(\pi\lambda/\beta)}e^{-i\lambda(\epsilon(x)-\mu)} = -\frac{\beta}{2}\frac{1}{1+\cosh(\beta(\epsilon-\mu))}. \tag{D-3}$$

To be more concrete, let us assume $\epsilon(x) = \hbar\omega_c x + \Delta E$ and $g(y)$ to be a polynomial of y. Then the following shows the idea behind the further steps of the calculation:

$$\begin{aligned}\int_0^\infty F(x)\cos(2\pi kx)dx &\approx \hbar\omega_c\text{Re}\Big\{\int_{-\infty}^\infty\int_{-\infty}^\infty \frac{d\lambda}{2\pi}\frac{\pi\lambda/\beta}{\sinh(\pi\lambda/\beta)}e^{i\lambda(\mu-\Delta E)}e^{-i\lambda\hbar\omega_c x}\Big[\int_0^x g(y)e^{2\pi iky}dy\Big]dx\Big\} = \\ &= \hbar\omega_c\text{Re}\Big\{\int_{-\infty}^\infty \frac{d\lambda}{2\pi}\frac{\pi\lambda/\beta}{\sinh(\pi\lambda/\beta)}e^{i\lambda(\mu-\Delta E)}\int_{-\infty}^\infty dx e^{-i\lambda\hbar\omega_c x}\Big[G(x)e^{2\pi ikx} + G_0\Big]\Big\} \approx \\ &\approx \text{Re}\Big\{\int_{-\infty}^\infty d\lambda\frac{\pi\lambda/\beta}{\sinh(\pi\lambda/\beta)}e^{i\lambda(\mu-\Delta E)}G\left(\frac{i\partial_\lambda}{\hbar\omega_c}\right)\delta\left(\frac{2\pi k}{\hbar\omega_c} - \lambda\right)\Big\} \approx \\ &\approx R_T(k)\underbrace{\text{Re}\Big\{e^{i\frac{2\pi k}{\hbar\omega_c}(\mu-\Delta E)}G\left(\frac{\mu-\Delta E}{\hbar\omega_c}\right)\Big\}}_{\text{zero temperature result}},\end{aligned} \tag{D-4}$$

where $\int_0^x g(y)e^{2\pi iky}dy = G(x)e^{2\pi ikx} + G_0$. Polynomial $G(x)$ is to be determined for each quantity of interest, and in this sketch we discarded G_0 as a non-oscillating contribution. For the last step we performed integration by parts, and approximated $-i\partial_\lambda \approx \mu - \Delta E + \mathcal{O}(k_B T_0)$.

1. Energy oscillations

Let us now apply the above for the system time average energy calculation. The time average system energy up to the second order in EF can be found using Eq.(3) of the main text and Eqs.(C-2,C-11) as follows:

$$\begin{aligned}\bar{E}(t) &= N_\phi\int_0^T \frac{dt}{T}\text{Tr}[\rho_S(t)H] = N_\phi\int_0^T \frac{dt}{T}\sum_{N=0}^\infty p(\epsilon_N^F)\langle\psi_N(t)|\,i\hbar d_t\,|\psi_N(t)\rangle \\ &= N_\phi\sum_{N=0}^\infty\Big[\sum_{l=-\infty}^\infty f_0(\epsilon_N^F + l\hbar\Omega)|\varphi_{N,l}|^2\Big]\Big[\sum_{l'=-\infty}^\infty[\epsilon_N^F + l'\hbar\Omega]|\varphi_{N,l'}|^2\Big]\end{aligned} \tag{D-5}$$

where we employed the Schrödinger equation $H(t)\,|\psi_N(t)\rangle = i\hbar d_t\,|\psi_N(t)\rangle$, and employed the normalised Floquet amplitudes as $|\varphi_{N,l}|^2 = \sum_b |c_{N,b}^l|^2/(\sum_{b,N}|c_{N,b}^l|^2)$, keeping terms up to the second order in powers of electric field. Using Eq.(C-13) and $\epsilon_N^F = (N+1/2)\hbar\omega_c + \Delta E$ we can rewrite the r.h.s. of the Eq.(D-5) as a functions of the N-th sate Floquet energy:

$$\sum_{l=-\infty}^\infty f_0(\epsilon_N^F + l\hbar\Omega)|\varphi_{N,l}|^2 = p(\epsilon_N^F), \qquad \sum_{l'=-\infty}^\infty[\epsilon_N^F + l'\hbar\Omega]|\varphi_{N,l'}|^2 = \varepsilon(\epsilon_N^F), \tag{D-6}$$

where up to the second order in powers of electric field, these functions are (see Eq.(C-14)):

$$p(\epsilon) = f_0(\epsilon) + \frac{\epsilon}{\hbar\omega_c} R_+ \Big[f_0(\epsilon + \hbar\Omega) + f_0(\epsilon - \hbar\Omega) - 2f_0(\epsilon)\Big] + R_- \Big[f_0(\epsilon + \hbar\Omega) - f_0(\epsilon - \hbar\Omega)\Big] \tag{D-7}$$

and $\varepsilon(\epsilon) = \epsilon + 2\hbar\Omega R_-$. Next, we employ the Poisson formula from Eq.(D-1) to get:

$$\bar{E}(B, \mu, T_0) = N_\phi \int_0^\infty dx p(\epsilon^F(x))\varepsilon(\epsilon^F(x)) + 2N_\phi \sum_{k=1}^{\infty} (-1)^k \int_0^\infty p(\epsilon^F(x))\varepsilon(\epsilon^F(x)) \cos(2\pi kx) dx \tag{D-8}$$

where $\epsilon^F(x) = x\hbar\omega_c + \Delta E$. Note, the form of the l.h.s. in Eq.(D-1) is evaluated at $x = N + 1/2$. The first term in the equation above contains the Landau orbital diamagnetic effect, while the second is related to the magnetic oscillations, which are of our interest and we focus on the second term only.

The integration of the magnetic oscillation term we perform using Eqs.(D-2-D-4), while neglecting terms of order $k_B T_0$ and assuming $\mu/\hbar\omega_c \gg 1$. Note that $p(\epsilon)$ in Eq.(D-7) is a sum of multiple terms of the form $f_0(x+a)g(x)$, thus according to Eqs.(D-2-D-4) this already guarantees oscillations at multiple frequencies. After applying the procedure to each of them, we obtain the following result for the magnetic oscillations:

$$\begin{aligned} \delta\bar{E}(B, \mu, T_0) - N_\phi \frac{\hbar\omega_c}{24} = N_\phi \sum_{l=\pm1,0} \sum_{k=1}^{\infty} \frac{\hbar\omega_c(-1)^k}{2k^2\pi^2} R_T(k) \bigg(\Big[a_l + 2b_l \frac{\mu_l}{\hbar\omega_c}\Big] \cos\Big(2\pi k \frac{\mu_l}{\hbar\omega_c}\Big) \\ + \Big[2\pi k \Big(a_l + b_l \frac{\mu_l}{\hbar\omega_c}\Big) \frac{\mu_l}{\hbar\omega_c}\Big] \sin\Big(2\pi k \frac{\mu_l}{\hbar\omega_c}\Big) \bigg), \end{aligned} \tag{D-9}$$

where $a_1 = -a_{-1} = R_-, a_0 = 1$, $b_1 = b_{-1} = R_+, b_0 = -2R_+$ and $\mu_\eta = \mu - \eta\hbar\Omega - \Delta E$. The R_T factor comes directly from the Fourier representation of the derivative of the Fermi-Dirac function in the Eq.(D-3), while the oscillations are the result of the integration in brackets of Eq.(D-2) over y.

The part of the oscillations in Eq.(D-9) proportional to the sin arise because the bath does not conserve particle number whereas the part proportional to cos is the free energy oscillations discussed in the main text. One can also rewrite the oscillations in terms of the Fermi surface ratio to the magnetic field by employing:

$$\mu = \frac{\hbar^2\pi k_F^2}{2\pi m} = \frac{\hbar^2 S}{2\pi m}, \qquad \omega_c = \frac{eB}{m}, \qquad 2\pi \frac{\mu}{\hbar\omega_c} = \frac{\hbar S}{eB}, \tag{D-10}$$

where S is the area of the main Fermi surface and in the main text we adopted $e = \hbar = c = 1$.

2. Floquet free energy oscillations

In this subsection, we provide some details on the derivation of the Floquet free energy oscillations, discussed in the main text. Using the definition Eq.(6) of the main text, Eq.(D-1) and $|\varphi_{l,N}|^2 = |\varphi_l(\epsilon_N^F)|^2$, we rewrite the Floquet free energy as:

$$\begin{aligned} -\frac{G}{N_\phi k_B T_0} = \sum_{l=-\infty}^{\infty} \int_0^\infty dx |\varphi_l(\epsilon^F(x))|^2 \log\Big[1 + e^{-\beta(\epsilon^F(x) + l\Omega - \mu)}\Big] + \\ + 2\sum_{l=-\infty}^{\infty} \sum_{k=1}^{\infty} (-1)^k \int_0^\infty |\varphi_l(\epsilon^F(x))|^2 \log\Big[1 + e^{-\beta(\epsilon^F(x) + l\Omega - \mu)}\Big] \cos(2\pi kx) dx. \end{aligned} \tag{D-11}$$

For the next step, we will keep the generality of the derivation. We perform a similar protocol shown in Eq.(D-2), yet here we integrate by parts twice, obtaining:

$$\frac{\delta G}{N_\phi k_B T_0} = 2\beta \sum_{l=-\infty}^{\infty} \sum_{k=1}^{\infty} (-1)^k \int_0^\infty \frac{df_0(\epsilon^F(x) + l\Omega)}{dx} \left\{ \int_0^x dz \frac{d\epsilon^F(z)}{dz} \left[\int_0^z |\varphi_l(\epsilon^F(y))|^2 \cos(2\pi ky) dy \right] \right\} dx. \tag{D-12}$$

Next, we move to the perturbative consideration, which allows us to simplify the above, obtaining:

$$\frac{\delta G}{N_\phi k_B T_0} \approx 2\beta\hbar\omega_c \sum_{l=-1}^{1} \sum_{k=1}^{\infty} (-1)^k \int_0^\infty \frac{df_0(\epsilon^F(x) + l\Omega)}{dx} \left\{ \int_0^x dz \left[\int_0^z |\varphi_l(\epsilon^F(y))|^2 \cos(2\pi ky) dy \right] \right\} dx, \tag{D-13}$$

which by using $\int_0^z dx \int_0^x f(y)dy = \int_0^z (z-y)f(y)dy$ becomes:

$$\frac{\delta G}{N_\phi k_B T_0} \approx 2\beta\hbar\omega_c \sum_{l=-1}^{1}\sum_{k=1}^{\infty}(-1)^k \int_0^\infty \frac{df_0(\epsilon^F(x)+l\Omega)}{dx}\left[\int_0^x (x-y)|\varphi_l(\epsilon^F(y))|^2\cos(2\pi k y)dy\right]dx. \tag{D-14}$$

Finally, by discarding contributions of order $k_B T_0$, the oscillations are found as:

$$\delta G \approx N_\phi \sum_{l=\pm1,0}\sum_{k=1}^{\infty}\frac{\hbar\omega_c}{2k^2\pi^2}(-1)^k R_T(k)\left(\delta_{l,0}+b_l\frac{\mu_l}{\hbar\omega_c}\right)\cos\left(2\pi k\frac{\mu_l}{\hbar\omega_c}\right), \tag{D-15}$$

which is the result reported in the main text.

3. Particle number oscillations

In this subsection we provide some details on the derivation of the DoS oscillation. We start from the particle number calculation that by definition is:

$$n(B,\mu,T_0) = \frac{1}{2\pi l_c^2}\sum_{N=0}^{\infty} p(\epsilon^F(N+1/2)) = \frac{1}{2\pi l_c^2}\int_0^\infty dx p(\epsilon^F(x)) + \frac{1}{\pi l_c^2}\sum_{k=1}^{\infty}(-1)^k\int_0^\infty p(\epsilon^F(x))\cos(2\pi k x)dx \tag{D-16}$$

and perform similar calculation as in the previous subsection obtaining the following oscillation part of the particle number:

$$\delta n(B,\mu,T_0) = \frac{1}{\pi l_c^2}\sum_{l=\pm1,0}\sum_{k=1}^{\infty}\frac{(-1)^k}{4k^2\pi^2}R_T(k)\left[b_l\cos\left(2\pi k\frac{\mu_l}{\hbar\omega_c}\right)+2\pi k\left(a_l+b_l\frac{\mu_\eta}{\hbar\omega_c}\right)\sin\left(2\pi k\frac{\mu_l}{\hbar\omega_c}\right)\right], \tag{D-17}$$

which allows us to immediately find the oscillating part of DoS as:

$$\begin{aligned}\delta\nu &= \lim_{T_0\to 0}\frac{\partial\delta n}{\partial\mu} \approx \frac{1}{\pi l_c^2\hbar\omega_c}\sum_{l=\pm1,0}\sum_{k=1}^{\infty}(-1)^k R_T(k)\left(a_l+b_l\frac{\mu_l}{\hbar\omega_c}\right)\cos\left(2\pi k\frac{\mu_l}{\hbar\omega_c}\right)\\ &\approx \frac{2}{\hbar\omega_c l_c^2}\sum_{k=1}^{\infty}(-1)^k R_T(k)\left[1-4R_+\frac{\mu}{\hbar\omega_c}\sin^2\left(\pi k\frac{\Omega}{\omega_c}\right)\right]\cos\left(2\pi k\frac{\mu}{\hbar\omega_c}\right),\end{aligned} \tag{D-18}$$

where we kept the leading term in $\mu_l/(\hbar\omega_c)$. Note, for the above summation to converge, one has to assume arbitrarily small, but non-zero temperature of the bath.

Appendix E: Non-equilibrium specific heat at fixed particle number

In this section we aim to derive the non-equilibrium specific heat at a fixed particle number, by defining it as the derivative of the time-averaged energy of the system with respect to temperature. This derivation follows Eq. (10) mentioned in the main text.

First, we establish the relationship between the chemical potential μ and the temperature T_0, keeping the particle number constant. For the Floquet case, following Eq. (10) in the main text, the total particle number is expressed as a function of both temperature and chemical potential:

$$n_0 = \sum_{l=-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_0(\epsilon)\nu_l(\epsilon)d\epsilon = \sum_{l=-\infty}^{+\infty}\int_{-\infty}^{\mu}\nu_l(\epsilon)d\epsilon + \frac{\pi^2}{6}(k_B T_0)^2\sum_{l=-\infty}^{+\infty}\nu_l'(\mu) + O({T_0}^4) \tag{E-1}$$

In the equation above, we have used the Sommerfeld expansion at low temperatures:

$$\int_{-\infty}^{\infty}\frac{g(\varepsilon)}{e^{(\varepsilon-\mu)/k_B T_0}+1}d\varepsilon = \int_{-\infty}^{\mu} g(\varepsilon)d\varepsilon + \frac{\pi^2}{6}(k_B T_0)^2 g'(\mu) + O(T_0)^4 \tag{E-2}$$

By recasting $\mu = \mu_0 + \delta\mu$, and expanding the right-hand side of Eq. (E-1) up to the first order of $\delta\mu$, we can deduce the relation between μ and T_0 at low temperatures:

$$\mu \approx \mu_0 - \frac{\pi^2}{6}(k_B T_0)^2 \frac{\sum_{l=-\infty}^{+\infty} \nu_l'(\mu_0)}{\sum_{l=-\infty}^{+\infty} \nu_l(\mu_0)}. \tag{E-3}$$

Subsequently, we calculate the non-equilibrium specific heat at a fixed particle number. In the Floquet case, the averaged total energy is given by:

$$\bar{E}(\mu, T_0) = \sum_{l=-\infty}^{+\infty} \int \frac{d\mathbf{k}}{(2\pi)^d} \left|\varphi_{\mathbf{k},l}\right|^2 f_0(\epsilon_{\mathbf{k}}^F + l\Omega)\epsilon_{\mathbf{k}}^F = \sum_{l=-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\epsilon f_0(\epsilon)\nu_l(\epsilon)(\epsilon - l\Omega). \tag{E-4}$$

By again using the Sommerfeld expansion at low temperature, we obtain

$$\bar{E}(\mu, T_0) = \sum_{l=-\infty}^{+\infty} \left[\int_{-\infty}^{\mu} (\epsilon - l\Omega)\nu_l(\epsilon) d\epsilon + \frac{\pi^2}{6}(k_B T_0)^2 \Big(\frac{\partial[(\epsilon - l\Omega)\nu_l(\epsilon)]}{\partial \epsilon} \Big)_{\epsilon=\mu} + O({T_0}^4) \right] \tag{E-5}$$

We then make use of the chain rule

$$C_V = \frac{\partial \bar{E}(\mu, T_0)}{\partial T_0} = \frac{\partial \bar{E}(\mu, T_0)}{\partial \mu} \frac{\partial \mu}{\partial T_0} + \frac{\partial \bar{E}(\mu, T_0)}{\partial T_0}, \tag{E-6}$$

then substituting $\partial\mu/\partial T_0$ from Eq. (E-3) into the equation above, we obtain the non-equilibrium specific heat, C_V, at fixed particle number as follows:

$$C_V = \frac{\pi^2}{3} k_B T_0 \sum_{l=-\infty}^{+\infty} \nu_l(\mu_0) + \frac{\pi^2}{3} k_B T_0 \frac{\Omega}{\sum_l \nu_l(\mu_0)} \sum_{l_2 l_1} (l_2 - l_1)\nu_{l_1}(\mu_0)\nu_{l_2}'(\mu_0), \tag{E-7}$$

which is the Eq. (14) shown in the main text.

Appendix F: Floquet amplitudes and van-Hove singularities in square lattice tight-binding Models

This section provides a discussion on the Floquet amplitude in the context of tight-binding models represented on a square lattice.

1. Harmonics of the periodic energy

We begin by considering a tight-binding model on a square lattice with a lattice constant $a = 1$, described by the dispersion relation

$$\epsilon_{\mathbf{k}} = -2t\cos(k_x) - 2t\cos(k_y). \tag{F-1}$$

We introduce a monochromatic AC driving expressed as

$$\mathbf{A}(t) = [A_x \sin(\Omega t + \phi_x), A_y \sin(\Omega t + \phi_y)]. \tag{F-2}$$

As a result, the dispersion relation of the system evolves periodically as

$$\epsilon_{\mathbf{k}} \to \epsilon_{\mathbf{k}}(t) = -2t\cos[k_x - A_x \sin(\Omega t + \phi_x)] - 2t\cos[k_y - A_y \sin(\Omega t + \phi_y)]. \tag{F-3}$$

Applying the Jacobi-Anger expansion allows us to derive the harmonics of the periodic energy as follows:

$$\begin{aligned} \epsilon_{\mathbf{k}}^{(l)} = \int_0^T \frac{dt}{T} \epsilon_{\mathbf{k}}(t) \exp(+il\Omega t) = 2t\Big[&- \frac{e^{-il(\phi_x+\pi)}}{2} J_{+l}(-A_x) e^{+ik_x} - \frac{e^{-il(\phi_x+\pi)}}{2} J_{+l}(+A_x) e^{-ik_x} \\ &- \frac{e^{-il(\phi_y+\pi)}}{2} J_{+l}(-A_y) e^{+ik_y} - \frac{e^{-il(\phi_y+\pi)}}{2} J_{+l}(+A_y) e^{-ik_y} \Big]. \end{aligned} \tag{F-4}$$

Consequently, the Floquet energy can be calculated as

$$\epsilon_{\mathbf{k}}^{(0)} = -2t\cos(k_x)J_0(A_x) - 2t\cos(k_y)J_0(A_y). \tag{F-5}$$

And we pay special attention to $\mathbf{k}$ points at $\mathbf{X} = (\pi, 0)$ and $\mathbf{Y} = (0, \pi)$ within the Brillouin zone, for which we obtain the following expressions

$$\begin{aligned} \epsilon_{\mathbf{X}}^{(l)} &= \frac{1+(-1)^l}{2}\,[+e^{-il(\phi_x+\pi)}J_{+l}(+A_x) - e^{-il(\phi_y+\pi)}J_{+l}(+A_y)], \\ \epsilon_{\mathbf{Y}}^{(l)} &= \frac{1+(-1)^l}{2}\,[-e^{-il(\phi_x+\pi)}J_{+l}(+A_x) + e^{-il(\phi_y+\pi)}J_{+l}(+A_y)]. \end{aligned} \tag{F-6}$$

From the above it follows that

$$\epsilon_{\mathbf{X}}^{(l)} = -\epsilon_{\mathbf{Y}}^{(l)} = \begin{cases} 0 & l \text{ is odd} \\ \text{non-zero} & l \text{ is even} \end{cases} \tag{F-7}$$

2. Amplitudes of Floquet harmonics

We now consider the amplitudes of Floquet harmonics, as expressed by the following equations:

$$\begin{aligned} \varphi_{\mathbf{k},l} &= \frac{1}{T}\int_0^T dt\Big[\exp(+il\Omega t) \times \exp\Big(-i\int_0^t dt'[\epsilon_{\mathbf{k}}(t') - \epsilon_{\mathbf{k}}^F]\Big)\Big] \\ &= \exp\Big(-\sum_{\substack{l_1=-\infty \\ l_1\neq 0}}^{+\infty}\frac{\epsilon_{\mathbf{k}}^{(l_1)}}{l_1\Omega}\Big) \times \int_0^T \frac{dt}{T}\exp\Big(\sum_{\substack{l_1=-\infty \\ l_1\neq 0}}^{+\infty}\frac{\epsilon_{\mathbf{k}}^{(l_1)}e^{-il_1\Omega t}}{l_1\Omega} + il\Omega t\Big). \end{aligned} \tag{F-8}$$

Note that the phase factor $-\sum_{l_1\neq 0}\epsilon_{\mathbf{k}}^{(l_1)}/(l_1\Omega)$ is purely imaginary due to the fact that $\epsilon_{\mathbf{k}}^{(l_1)} = [\epsilon_{\mathbf{k}}^{(-l_1)}]^*$ which can be seen from Eq. (F-4). To compute $|\varphi_{\mathbf{k}}^{(l)}|^2$, we can disregard the global phase factor outside the integral and focus on the integral within it:

$$\phi_{\mathbf{k},l} = \int_0^T \frac{dt}{T}\exp\Big(\sum_{\substack{l_1=-\infty \\ l_1\neq 0}}^{+\infty}\frac{\epsilon_{\mathbf{k}}^{(l_1)}e^{-il_1\Omega t}}{l_1\Omega} + il\Omega t\Big), \quad |\phi_{\mathbf{k},l}|^2 = |\varphi_{\mathbf{k},l}|^2. \tag{F-9}$$

We perform a change of variable, $z = \exp(i\Omega t)$:

$$\phi_{\mathbf{k},l} = \frac{1}{2\pi i}\oint_{\arg[z]=0}^{\arg[z]=2\pi}\prod_{\substack{l_1=-\infty \\ l_1\neq 0}}^{+\infty}\exp\Big(\frac{\epsilon_{\mathbf{k}}^{(l_1)}z^{-l_1}}{l_1\Omega}\Big)z^l\frac{dz}{z}, \tag{F-10}$$

We then express this equation by expanding the exponential functions using Taylor series and performing Cauchy's residue theorem:

$$\phi_{\mathbf{k},l} = \delta_{0,l} + \sum_{n_1\times l_1=l}\frac{1}{n_1!}\Big(\frac{\epsilon_{\mathbf{k}}^{(l_1)}}{l_1\Omega}\Big)^{n_1} + \sum_{n_1\times l_1+n_2\times l_2=l}\frac{1}{n_1!}\Big(\frac{\epsilon_{\mathbf{k}}^{(l_1)}}{l_1\Omega}\Big)^{n_1}\frac{1}{n_2!}\Big(\frac{\epsilon_{\mathbf{k}}^{(l_2)}}{l_2\Omega}\Big)^{n_2} + \cdots . \qquad (l_{1,2,3,\cdots}\neq 0;\quad n_{1,2,3,\cdots}\geq 1) \tag{F-11}$$

Building upon equation (F-11), we deduce the following: if l is odd, then

$$n_1\times l_1 + n_2\times l_2 + n_3\times l_3 + \cdots \in \text{ odd} \quad \rightarrow \quad \text{at least odd numbers of } (n_i\in\text{odd}, l_i\in\text{odd}) \text{ pairs} \tag{F-12}$$

We then incorporate the conclusion from Eq. (F-7) which states that $\epsilon_{\mathbf{X}}^{(l)} = -\epsilon_{\mathbf{Y}}^{(l)} = 0$ when l is odd, into Eq. (F-11). This yields:

$$\phi_{\mathbf{X},l} = \phi_{\mathbf{Y},l} = 0, \qquad l\in\text{odd}. \tag{F-13}$$

This result confirms that the contributions associated with the odd harmonics have vanishing weight at these special momenta, namely $\varphi_{(0,\pi),l} = \varphi_{(\pi,0),l} = 0$ for odd l.

3. Floquet van-Hove singularities

Let us first consider the DoS for equilibrium band without driving for the dispersion from Eq. (F-1). For this model, the density of state is given by

$$\nu(\mu) = \frac{4}{|2t|}\int_0^{+\pi}\frac{dk_x}{2\pi}\int_0^{+\pi}\frac{dk_y}{2\pi}\delta(-\cos(k_x)-\cos(k_y)-\tilde{\mu}), \quad \tilde{\mu}=\mu/2t. \tag{F-14}$$

By performing a change of variables, $u=-\cos(k_x)$ and $v=-\cos(k_y)$, we obtain

$$\nu(\mu) = \frac{2}{|t|}\frac{1}{(2\pi)^2}\int_{-1}^{+1}du\int_{-1}^{+1}dv\frac{1}{\sqrt{1-u^2}\sqrt{1-v^2}}\delta(u+v-\mu). \tag{F-15}$$

Notably, the singularity in the above equation originates from the singularities of the integrand at

$$u=+1, v=-1 \text{ or } u=-1, v=+1 \tag{F-16}$$

or equivalently, from the following special $\mathbf{k}$ points:

$$\mathbf{X}=(\pi,0) \text{ or } \mathbf{Y}=(0,\pi), \tag{F-17}$$

leading to divergence of the integral at $\tilde{\mu}=u+v=0$ at half filling.

We now proceed to introduce the driving as per Eqs. (F-2) and (F-3). Following the definitions given in Eqs. (4) and (10) in the main text, we obtain the following relationships:

$$\nu(\mu) = \sum_{l=-\infty}^{+\infty}\nu_l(\mu_l), \quad \mu_l=\mu-l\Omega, \tag{F-18}$$

and

$$\nu_l(\mu_l) = \int_{-\pi}^{+\pi}\frac{dk_x}{2\pi}\int_{-\pi}^{+\pi}\frac{dk_y}{2\pi}|\phi_{\mathbf{k}}^{(l)}|^2\delta(\epsilon_{\mathbf{k}}^{(0)}-\mu_l) = \sum_{\eta_x,\eta_y=\pm1}\nu_l^{(\eta_x,\eta_y)}(\mu_l). \tag{F-19}$$

Here, $\nu_l^{(\eta_x,\eta_y)}(\mu_l)$ correspond to different patches of the Brillouin zone, which we define as follows:

$$\begin{aligned}
\nu_l^{(-1,-1)}(\mu_l) &= \int_{-\pi}^{0}\frac{dk_x}{2\pi}\int_{-\pi}^{0}\frac{dk_y}{2\pi}|\phi_{\mathbf{k}}^{(l)}|^2\delta(\epsilon_{\mathbf{k}}^{(0)}-\mu_l), \quad \nu_l^{(-1,+1)}(\mu_l) = \int_{-\pi}^{0}\frac{dk_x}{2\pi}\int_{0}^{+\pi}\frac{dk_y}{2\pi}|\phi_{\mathbf{k}}^{(l)}|^2\delta(\epsilon_{\mathbf{k}}^{(0)}-\mu_l),\\
\nu_l^{(+1,-1)}(\mu_l) &= \int_{0}^{+\pi}\frac{dk_x}{2\pi}\int_{-\pi}^{0}\frac{dk_y}{2\pi}|\phi_{\mathbf{k}}^{(l)}|^2\delta(\epsilon_{\mathbf{k}}^{(0)}-\mu_l), \quad \nu_l^{(+1,+1)}(\mu_l) = \int_{0}^{+\pi}\frac{dk_x}{2\pi}\int_{0}^{+\pi}\frac{dk_y}{2\pi}|\phi_{\mathbf{k}}^{(l)}|^2\delta(\epsilon_{\mathbf{k}}^{(0)}-\mu_l),
\end{aligned} \tag{F-20}$$

where $\eta_{x,y}=\pm1$ indicates four different patches of the Brillouin zone. This division allows for a one-to-one change of variables given by

$$u=-\cos(k_x)J_0(A_x), \qquad v=-\cos(k_y)J_0(A_y), \tag{F-21}$$

which subsequently results in Jacobians on different patches as shown below:

$$\left|\frac{\partial(k_x,k_y)}{\partial(u,v)}\right|^{(\eta_x,\eta_y)} = \frac{|\eta_x\eta_y|}{\sqrt{J_0^2(A_x)-u^2}\sqrt{J_0^2(A_y)-v^2}}, \quad \eta_{x,y}=\pm1. \tag{F-22}$$

By incorporating these transformations, we can express $\nu_l^{(\eta_x,\eta_y)}(\mu_l)$ as

$$\nu_l^{(\eta_x,\eta_y)}(\mu_l) = \frac{2}{|t|}\frac{1}{(2\pi)^2}\int_{-|J_0(A_x)|}^{+|J_0(A_x)|}du\int_{-|J_0(A_y)|}^{+|J_0(A_y)|}dv\frac{|\phi_l^{\eta_x,\eta_y}(u,v)|^2}{\sqrt{J_0^2(A_x)-u^2}\sqrt{J_0^2(A_y)-v^2}}\delta(u+v-\tilde{\mu}_l). \tag{F-23}$$

Here, $\phi_l^{\eta_x,\eta_y}(u,v)$ represents the transformation from $\phi_{\mathbf{k},l}$ on each of the four patches. This means that we need to choose different signs for $\sin(k_x)$ and $\sin(k_y)$ based on the values of η_x and η_y.

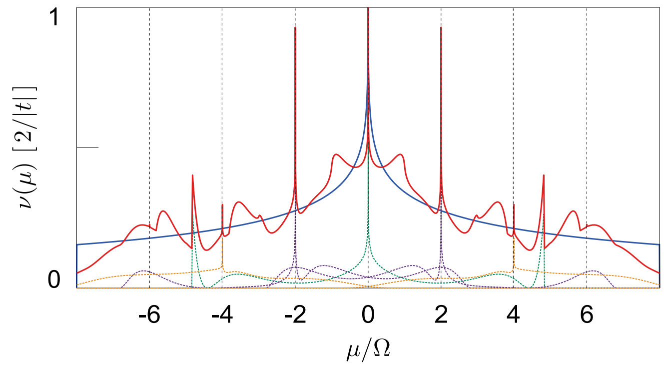

FIG. F-4. DoS for the non-driven (blue) and driven (red) square lattice model from Eq.(10). Dashed lines are the additional van-hove singularities from Eq.(12). Green, purple, and orange lines are contributions making the red-line coming from $l = 0, \pm 2, \pm 4$ harmonics respectively.

Examining Eq. (F-23), it becomes evident that, unlike in the non-driven case, each of the $\nu_l(\mu_l)$ can potentially have two separate van-Hove singularities that could originate from the singularities of the integrand at

$$u = +J_0(A_x), v = -J_0(A_y) \text{ or } u = -J_0(A_x), v = +J_0(A_y), \tag{F-24}$$

or, equivalently, from the special $\mathbf{k}$ points

$$\mathbf{X} = (\pi, 0) \text{ or } \mathbf{Y} = (0, \pi), \tag{F-25}$$

which create van-Hove singularities when the conditions

$$\tilde{\mu}_l = +J_0(A_x) - J_0(A_y) \text{ or } \tilde{\mu}_l = -J_0(A_x) + J_0(A_y) \tag{F-26}$$

are met, in which $\tilde{\mu}_l = (\mu - l\Omega)/(2t)$. Moreover, we also observe that:

(i) when $A_x = A_y$, the two distinct singularities merge into a single singularity;

(ii) as demonstrated in Eq. (F-13), for odd values of l, the conditions $\phi_{\mathbf{X},l} = \phi_{\mathbf{Y},l} = 0$ and therefore $|\varphi_{\mathbf{X},l}|^2 = |\varphi_{\mathbf{Y},l}|^2 = 0$ hold, which prevent the appearance of van-Hove singularities for $\nu_l(\mu_l)$ when l is odd in the current model.;

(iii) Equations (F-4), (F-8), and (F-23), with particular emphasis on Eq.(F-9), offer a non-perturbative expression for numerically determining the density of states. The convergence stems from the relationship $\epsilon_{\mathbf{k}}^{(l)} \sim J_l(A_{x,y})$, which decays rapidly as l increases for a given $A_{x,y}$. For our numerical evaluations concerning Eq.(F-9), we sum over values of l_1 ranging from -100 to $+100$, ensuring well-converged results [see Fig. F-4, an extended plot of Fig. 3(a) in the main text].

Appendix G: Pair correlation function of the Floquet Fermi Liquid

The pair correlation function measures the probability density of finding a particle at position $\mathbf{r}_1$ given that another particle is at position $\mathbf{r}_2$, and it is therefore an important characterization of the correlations in a fluid. This correlation function can be obtained as follows [44]

$$g(\mathbf{r}_1, \mathbf{r}_2, t) = \frac{\langle n(\mathbf{r}_1, t) n(\mathbf{r}_2, t)\rangle}{\langle n(\mathbf{r}_1, t)\rangle\langle n(\mathbf{r}_2, t)\rangle} - \frac{\delta(\mathbf{r}_2 - \mathbf{r}_1)}{\langle n(\mathbf{r}_1, t)\rangle} = 1 - \frac{\langle c^\dagger_{\mathbf{r}_1}(t) c_{\mathbf{r}_2}(t)\rangle\langle c^\dagger_{\mathbf{r}_2}(t) c_{\mathbf{r}_1}(t)\rangle}{n_0^2}, \tag{G-1}$$

where we applied the Wick's theorem, and used the fact that the system is translational invariant, i.e., $\langle n(\mathbf{r}_1, t_1)\rangle = \langle n(\mathbf{r}_2, t_2)\rangle = n_0$ is the averaged uniform density. For our single-band model of interest, the creation/annihilation operators in Eq. (G-1) have the following expansion in terms of plane waves:

$$c^\dagger_{\mathbf{r}}(t) = \frac{1}{\sqrt{A}} \sum_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{r}} [\psi^F_{\mathbf{k}}(t)]^* c^\dagger_{\mathbf{k}}, \tag{G-2}$$

where $\psi^F_{\mathbf{k}}(t)$ is the Floquet wave function defined in Appendix B (replacing the generic subscript a to $\mathbf{k}$ in a one-band model), then in the the Floquet Fermi Liquid steady state, we have $\langle c^\dagger_{\mathbf{k}'} c_{\mathbf{k}}\rangle = \delta_{\mathbf{k}'\mathbf{k}} f_{\mathbf{k}}$, and denoting $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, we obtain for Eq. (G-1)

$$g(\mathbf{r}) = 1 - \left| \frac{1}{n_0} \sum_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{r}} f_{\mathbf{k}} \right|^2. \tag{G-3}$$

In the above expression $f_{\mathbf{k}}$ is the steady state Fermi Dirac staircase occupation of the Floquet band.

In a conventional equilibrium Fermi liquid at $T_0 = 0$, $f_{\mathbf{k}} \to \Theta(k_F - |\mathbf{k}|)$. Therefore, as a consequence of the discontinuity of occupations at the Fermi surface the pair correlation function $g(\mathbf{r})$ exhibits long-range oscillatory behavior, which is governed by the Fermi wave vector k_F and the system's dimensionality [44]. This is closely related to the phenomenon of Friedel oscillations, except that here we do not have an external impurity perturbing the fluid, but instead the oscillations are self-consistently contained in the probability density of finding particles at certain distance from any given particle. As we will show now, the Floquet Fermi liquid at $T_0 = 0$ displays beating patterns in the long-range oscillations arising from its multiple Floquet Fermi surfaces.

For concreteness let us focus on a 1D band with dispersion in equilibrium given by $\epsilon(k) = (\hbar k)^2/2m$, however similar conclusions will hold in higher dimensional Floquet Fermi Liquids. This becomes $\epsilon(k) \to \epsilon(k - eA_0 \sin \Omega t/\hbar)$ under the periodic drive. Employing Eqs. (4) and (5), the occupation of the Floquet Fermi liquid at $T_0 = 0$ is given by:

$$f_k = \sum_l |\varphi_{k,l}|^2 \Theta(k_F^{(l)} - |k|). \tag{G-4}$$

Here, $k_F^{(l)}$ is the Floquet Fermi wave vector for the l-th Floquet band, given by

$$\epsilon^F(k_F^{(l)}) = \frac{(\hbar k_F^{(l)})^2}{2m} + \frac{1}{2}\frac{(eA_0)^2}{2m} = \mu - l\Omega. \tag{G-5}$$

The asymptotic behavior at large distances of the pair correlation function will be dominated by the non-analytic discontinuities of the Fermi-Dirac staircase occupation function. To see this, we decompose the occupation into its discontinuous and continuous parts as follows:

$$f_k = f_{\text{conti}}(k) + \sum_l |\varphi_l(k_F^{(l)})|^2 \Theta(k_F^{(l)} - |k|). \tag{G-6}$$

Here $f_{\text{conti}}(k)$ denotes the continuous part of f_k. Therefore we have:

$$\begin{aligned} \frac{1}{n_0}\sum_k e^{-ikr} f_k &= \frac{\tilde{f}_{\text{short}}(r)}{n_0} + \frac{1}{n_0}\sum_l \frac{|\varphi_l(k_F^{(l)})|^2}{2\pi} \int_{-k_F^{(l)}}^{+k_F^{(l)}} e^{-ikr} dk \\ &= \frac{\tilde{f}_{\text{short}}(r)}{n_0} + \sum_l |\varphi_l(k_F^{(l)})|^2 \frac{\sin(k_F^{(l)} r)}{k_F r}, \end{aligned} \tag{G-7}$$

where the identity $n_0 = \sum_k f_k = k_F/\pi$ has been used. Here, $\tilde{f}_{\text{short}}(r)$ is the Fourier transformation of the continuous $f_{\text{conti}}(k)$ and represents the short-ranged component in r (compared to those terms transformed from Theta functions). Subsequently, by incorporating Eq. (G-7) into Eq. (G-3), the following expression for $g(r)$ is obtained:

$$g(r) = 1 - \left| \frac{\tilde{f}_{\text{short}}(r)}{n_0} + \sum_l |\varphi_l(k_F^{(l)})|^2 \frac{\sin(k_F^{(l)} r)}{k_F r} \right|^2. \tag{G-8}$$

As r becomes significantly large, beating patterns emerge due to presence of several Floquet Fermi surfaces with different Floquet Fermi wave vectors $k_F^{(l)}$:

$$\lim_{r\to+\infty} r^2[1 - g(r)] = \left| \sum_l |\varphi_l(k_F^{(l)})|^2 \frac{\sin(k_F^{(l)} r)}{k_F} \right|^2. \tag{G-9}$$

We demonstrate this behavior directly by plotting a representative example of the pair correlation functions vs distance in Fig. G-5. The lower panel in Fig. G-5(b) shows the correlations at large distances computed with the full Eq. (G-8) and the asymptotic forms Eq. (G-9). We see the clear beating pattern and also that the asymptotic and exact expressions have nearly perfect agreement at long distances.

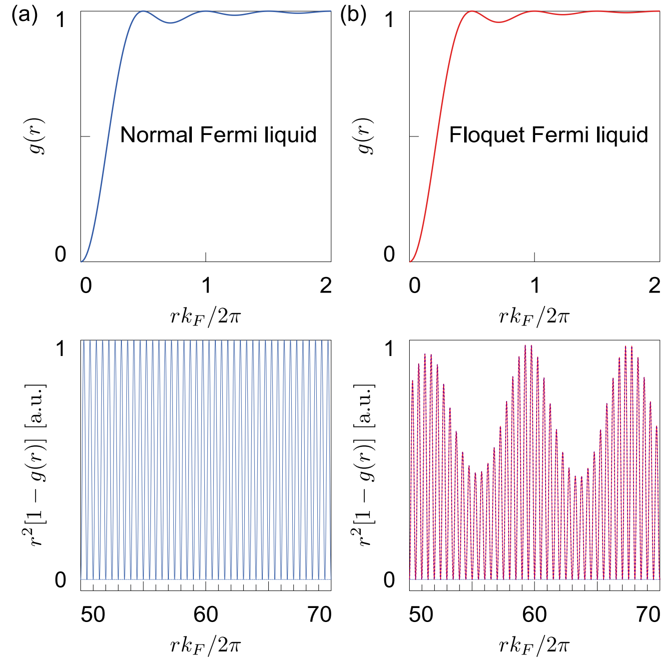

FIG. G-5. Pair correlation function $g(r)$ and the re-scaled long-range oscillations $r^2[1-g(r)]$ for a 1D parabolic Fermi liquid (a, blue solid lines), and the corresponding 1D Floquet Fermi liquid (b, red solid lines) calculated using Eqs. (G-3) and (G-4). The solid red line is the full exact result from Eq. (G-8) and the dotted purple line in (b) is calculated using the asymptotic expansion at large distance Eq. (G-9). We see that the agreement is essentially perfect at large distances. Parameters used: $(\hbar k_F)^2/2m = \epsilon_F$, $\hbar\Omega/\epsilon_F = 1/4$, $eA_0/\hbar k_F = 1/13$.

Appendix H: Un-equal time correlations

The main-body of our paper is focused on observables that can be computed from the equal-time one-body density matrix, such as the specific heat or density of states. However, it is also of interest to consider correlation functions at un-equal times, which allow to access various experimentally relevant quantities, in particular the spectra of noise and fluctuations of the system in contact with the bath.

As a concrete example of this we will consider in this appendix the un-equal time correlations of the density fluctuations, which are captured by the following correlator:

$$C(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2) = \langle n(\mathbf{r}_1, t_1) n(\mathbf{r}_2, t_2)\rangle - \langle n(\mathbf{r}_1, t_1)\rangle\langle n(\mathbf{r}_2, t_2)\rangle, \tag{H-1}$$

where the averages above are understood to be computed in the non-equilibrium Floquet steady state. Because the system has translational invariance $\langle n(\mathbf{r}_1, t_1)\rangle = \langle n(\mathbf{r}_2, t_2)\rangle = n_0$. The nontrivial part of $C(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2)$ is:

$$\langle n(\mathbf{r}_1, t_1) n(\mathbf{r}_2, t_2)\rangle = \langle c^\dagger_{\mathbf{r}_1}(t_1) c_{\mathbf{r}_1}(t_1) c^\dagger_{\mathbf{r}_2}(t_2) c_{\mathbf{r}_2}(t_2)\rangle. \tag{H-2}$$

Inserting Eq. (G-2) into Eq. (H-2) leads to

$$\begin{aligned}\langle c^\dagger_{\mathbf{r}_1}(t_1) c_{\mathbf{r}_1}(t_1) c^\dagger_{\mathbf{r}_2}(t_2) c_{\mathbf{r}_2}(t_2)\rangle = \frac{1}{A^2} \sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4} & e^{-i(\mathbf{k}_1-\mathbf{k}_2)\cdot\mathbf{r}_1 - i(\mathbf{k}_3-\mathbf{k}_4)\cdot\mathbf{r}_2} \\ & \times [\psi^F_{\mathbf{k}_1}(t_1)]^* \psi^F_{\mathbf{k}_2}(t_1) [\psi^F_{\mathbf{k}_3}(t_2)]^* \psi^F_{\mathbf{k}_4}(t_2) \langle c^\dagger_{\mathbf{k}_1} c_{\mathbf{k}_2} c^\dagger_{\mathbf{k}_3} c_{\mathbf{k}_4}\rangle.\end{aligned} \tag{H-3}$$

Because we are dealing with a system of non-interacting fermions, we can apply Wick's theorem $\langle c^\dagger_{\mathbf{k}_1} c_{\mathbf{k}_2} c^\dagger_{\mathbf{k}_3} c_{\mathbf{k}_4}\rangle = \langle c^\dagger_{\mathbf{k}_1} c_{\mathbf{k}_2}\rangle\langle c^\dagger_{\mathbf{k}_3} c_{\mathbf{k}_4}\rangle + \langle c^\dagger_{\mathbf{k}_1} c_{\mathbf{k}_4}\rangle\langle c_{\mathbf{k}_2} c^\dagger_{\mathbf{k}_3}\rangle$, taking the average [i.e., $\langle c^\dagger_a c_b\rangle = f_a \delta_{ab}$, $\langle c_a c^\dagger_b\rangle = (1-f_a)\delta_{ab}$] and the identity $\sum_l \langle \varphi_{a,l}|\varphi_{a,l}\rangle = 1$, we obtain

$$\begin{aligned}\langle c^\dagger_{\mathbf{r}_1}(t_1) c_{\mathbf{r}_1}(t_1) c^\dagger_{\mathbf{r}_2}(t_2) c_{\mathbf{r}_2}(t_2)\rangle = n_0^2 + \frac{1}{A^2} \sum_{\mathbf{k}_1,\mathbf{k}_2} & f_{\mathbf{k}_1}(1-f_{\mathbf{k}_2}) e^{-i(\mathbf{k}_1-\mathbf{k}_2)\cdot(\mathbf{r}_1-\mathbf{r}_2)} \\ & \times [\psi^F_{\mathbf{k}_1}(t_1)]^* \psi^F_{\mathbf{k}_2}(t_1) [\psi^F_{\mathbf{k}_2}(t_2)]^* \psi^F_{\mathbf{k}_1}(t_2)\end{aligned} \tag{H-4}$$

and the un-equal time correlation function is then

$$C(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2)=\frac{1}{A^2}\sum_{\mathbf{k}_1,\mathbf{k}_2}f_{\mathbf{k}_1}(1-f_{\mathbf{k}_2})e^{-i(\mathbf{k}_1-\mathbf{k}_2)\cdot(\mathbf{r}_1-\mathbf{r}_2)}[\psi^F_{\mathbf{k}_1}(t_1)]^*\psi^F_{\mathbf{k}_2}(t_1)[\psi^F_{\mathbf{k}_2}(t_2)]^*\psi^F_{\mathbf{k}_1}(t_2). \tag{H-5}$$

Its Fourier transform with respect to distance $\mathbf{r}_1-\mathbf{r}_2$, is then:

$$C(\mathbf{q};t_1,t_2)=\frac{1}{A}\sum_{\mathbf{k}}f_{\mathbf{k}+\mathbf{q}}(1-f_{\mathbf{k}})[\psi^F_{\mathbf{k}+\mathbf{q}}(t_1)]^*\psi^F_{\mathbf{k}}(t_1)[\psi^F_{\mathbf{k}}(t_2)]^*\psi^F_{\mathbf{k}+\mathbf{q}}(t_2). \tag{H-6}$$

In general because there is no time-translational invariance, the above correlation cannot be converted to a single correlation in frequency domain, and a two-frequency Fourier transform would be needed. As a special case, we can define a time averaged noise, by performing the Fourier transform with respect to the time difference and average over one period, as follows:

$$\begin{aligned}\bar{C}(\mathbf{q},\omega)&=\int_0^T\frac{\mathrm{d}t_2}{T}\left(\int_{-\infty}^{+\infty}\frac{\mathrm{d}t}{2\pi}C(\mathbf{q};t_2+t,t_2)e^{+i\omega t}\right)\\&=\frac{1}{A}\sum_l\sum_{\mathbf{k}}f_{\mathbf{k}+\mathbf{q}}(1-f_{\mathbf{k}})\Big|\sum_{l'}\varphi^*_{\mathbf{k},l}\varphi_{\mathbf{k}+\mathbf{q},l+l'}\Big|^2\delta(\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{k}+\mathbf{q}}-\omega+l\Omega),\end{aligned} \tag{H-7}$$

where we used the Floquet expansion $\psi^F_a(t)=\sum_l e^{-i(\epsilon^F_a+l\Omega)t}\varphi_{a,l}$.

In equilibrium, $\varphi_{\mathbf{k},l}=\delta_{l,0}$ and $\bar{C}(\mathbf{q},\omega)$ reduces to $(1/A)\sum_{\mathbf{k}}f_{\mathbf{k}+\mathbf{q}}(1-f_{\mathbf{k}})\delta(\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{k}+\mathbf{q}}-\omega)$, which is only non-zero over the region of the so-called electron–hole continuum and captures the spatio-temporal noise associated with density fluctuations for the conventional Fermi liquid [44]. Therefore we can see $\bar{C}(\mathbf{q},\omega)$ from Eq. (H-7) as the quantity that generalizes this result and characterizes the spectrum of noise associated with density fluctuations of the Floquet Fermi liquid.

Appendix I: Relation to the Keldysh formalism and two-time Green functions

We start from the Hamiltonian from Eq. (1) in the main text written in the second quantized form:

$$H_S(t)=\sum_{ab}c^\dagger_a H^{ab}_S(t)c_b,\quad H_B=\sum_{a,j}\varepsilon_j c^\dagger_{ja}c_{ja},\quad H_{BS}=V\sum_{a,j}c^\dagger_{ja}c_a=H^\dagger_{SB}. \tag{I-1}$$

The time evolution of fermionic operators in Heisenberg's picture:

$$c_a(t)=U_S(t,t_0)c_aU_S(t_0,t),\qquad i\partial_t c_a=[H_S(t),c_a]. \tag{I-2}$$

is governed by the following equations of motion for the system and bath operators respectively:

$$i\partial_t c^\dagger_a(t)=\sum_b H^{ab}_S(t)c_b+V\sum_j c_{ja},\quad i\partial_t c_{ja}=\varepsilon_j c_{ja}+Vc_a \tag{I-3}$$

After integrating out the second equation above for the bath destruction operators over time, and performing the same type of analysis but in second quantization notation from Appendix A and also detailed in [13, 32], the equation of motion for the system electronic operators in the presence of the featureless bath is:

$$i\partial_t c_a(t)=\sum_b[H^{ab}_S(t)-i\Gamma\delta^{ab}]c_b(t)+V\sum_j e^{-i\varepsilon_j(t-t_0)}c_{ja}(t_0) \tag{I-4}$$

where the solution of the above can be written as:

$$c_a(t)=e^{-\Gamma(t-t_0)}U^{ab}_S(t,t_0)c_b(t_0)-iV\sum_j e^{i\varepsilon_j t_0}\int_{t_0}^t e^{-\Gamma(t-t')-i\varepsilon_j t'}U^{ab}_S(t,t')c_{jb}(t_0)dt' \tag{I-5}$$

Where $U^{(ab)}_S(t,t_0)$ is the time evolution matrix of the system alone in the absence of the bath. From the above we can now compute the retarded Green function [45] for a bath with a constant density of states. In the limit $t_0\to-\infty$

it reads [46]:

$$
\begin{aligned}
iG^R_{ab}(t_1,t_2) &= \theta(t_1-t_2)\langle\{c_a(t_1),c^\dagger_b(t_2)\}\rangle \\
&= \theta(t_1-t_2)\Gamma\int_{-\infty}^{+\infty}\frac{\mathrm{d}\epsilon}{\pi}\left(\int_{-\infty}^{t_1}e^{-\Gamma(t_1-t')-i\epsilon t'}U_S(t_1,t')dt'\right)_{aa'}\left(\int_{-\infty}^{t_2}e^{-\Gamma(t_2-t')+i\epsilon t'}U_S(t',t_2)dt'\right)_{a'b} \\
&= \theta(t_1-t_2)\Gamma\int_{-\infty}^{+\infty}\frac{\mathrm{d}\epsilon}{\pi}\left[U_\Gamma(t_1,\epsilon)U^\dagger_\Gamma(t_2,\epsilon)\right]_{ab}.
\end{aligned}
\tag{I-6}
$$

As expected the retarded Green function contains the information of the propagation but no information about the occupations [46]; instead, the information occupations is contained in the lesser Green function [46], which we can compute from Eq. (I-5) assuming that at the past initial time the occupation of the bath is the Fermi-Dirac distribution $\langle c^\dagger_{ja}(-\infty)c_{ja}(-\infty)\rangle = f_0(\varepsilon_j)$:

$$
\begin{aligned}
iG^<_{ab}(t_1,t_2) &= \langle c^\dagger_b(t_2)c_a(t_1)\rangle \\
&= \Gamma\int_{-\infty}^{+\infty}\frac{\mathrm{d}\epsilon}{\pi}f_0(\epsilon)\left(\int_{-\infty}^{t_1}e^{-\Gamma(t_1-t')-i\epsilon t'}U_S(t_1,t')dt'\right)_{aa'}\left(\int_{-\infty}^{t_2}e^{-\Gamma(t_2-t')+i\epsilon t'}U_S(t',t_2)dt'\right)_{a'b} \\
&= \Gamma\int_{-\infty}^{+\infty}\frac{d\epsilon}{\pi}f_0(\epsilon)\left[U_\Gamma(t_1,\epsilon)U^\dagger_\Gamma(t_2,\epsilon)\right]_{ab}.
\end{aligned}
\tag{I-7}
$$

Comparing the above Eq. (I-7) and Eq. (A-8), we see that the equal time lesser Green function is identical to the density matrix $iG^<_{ab}(t,t) = \rho_S(t)$.

Apart from the above generic conclusion, we also show both Green functions explicitly in the limit of vanishing coupling to the bath. In both cases the integral over ϵ has the form:

$$
\Gamma\int_{-\infty}^{\infty}\frac{d\epsilon}{\pi}\frac{f_0(\epsilon)e^{-i\eta\epsilon}}{(\epsilon-A+i\Gamma)(\epsilon-B-i\Gamma)} = \frac{\Gamma}{B-A+2i\Gamma}\int_{-\infty}^{\infty}\frac{d\epsilon}{\pi}f_0(\epsilon)e^{-i\eta\epsilon}\left(\frac{1}{\epsilon-B-i\Gamma}-\frac{1}{\epsilon-A+i\Gamma}\right), \tag{I-8}
$$

where in case of the retarded Green functions one should replace $f_0 \to 1$. The integration over ϵ can be carried out exactly using the residue theorem. In the case of the lesser Green functions, the resulting Hypergeometric function is regular in $\Gamma \to 0$ and in a limit $\eta \to 0$ becomes a Polygamma function displayed in Eq. (B-6). The ratio $\Gamma/(B-A+2i\Gamma)$ in the clean limit reduces to $\delta_{AB}/(2i)$, which allows us to write:

$$
\lim_{\Gamma\to 0}\Gamma\int_{-\infty}^{+\infty}\frac{d\epsilon}{\pi}\frac{f_0(\epsilon)e^{-i\eta\epsilon}}{(\epsilon-A+i\Gamma)(\epsilon-B-i\Gamma)} = \delta_{AB}\lim_{\Gamma\to 0}\int_{-\infty}^{+\infty}\frac{d\epsilon}{\pi}f_0(\epsilon)e^{-i\eta\epsilon}\frac{\Gamma}{(\epsilon-A)^2+\Gamma^2} = \pi\delta_{AB}f_0(A)e^{-i\eta A}. \tag{I-9}
$$

Consequently we have for the lesser Green function:

$$
\begin{aligned}
&\lim_{\Gamma\to 0}\Gamma\int\frac{d\epsilon}{\pi}f_0(\epsilon)U_\Gamma(t_1,\epsilon)U^\dagger_\Gamma(t_2,\epsilon) \\
&= \lim_{\Gamma\to 0}\Gamma\int\frac{d\epsilon}{\pi}f_0(\epsilon)\sum_{a,b,l_1,l_2}|\varphi_{a,l_1}\rangle\langle\varphi_{b,l_2}|\,e^{i\Omega l_2 t_2-i\Omega l_1 t_1}\left(\sum_{l_3,l_4}e^{i\Omega l_3 t_1-i\Omega l_4 t_2}\frac{\langle\varphi_{a,l_3}|\,\varphi_{b,l_4}\rangle e^{-i\epsilon(t_1-t_2)}}{(\epsilon-\epsilon^F_a-\Omega l_3+i\Gamma)(\epsilon-\epsilon^F_b-\Omega l_4-i\Gamma)}\right) \\
&= \sum_a|\psi^F_a(t_1)\rangle\langle\psi^F_a(t_2)|\left(\sum_l f_0(\epsilon^F_a+l\Omega)\langle\varphi_{a,l}|\,\varphi_{a,l}\rangle\right).
\end{aligned}
$$

and we can conclude:

$$
\lim_{\Gamma\to 0}iG^R(t_1,t_2) = \theta(t_1-t_2)\sum_a|\psi^F_a(t_1)\rangle\langle\psi^F_a(t_2)|\,, \qquad \lim_{\Gamma\to 0}iG^<(t_1,t_2) = \sum_a f_a\,|\psi^F_a(t_1)\rangle\langle\psi^F_a(t_2)| \tag{I-10}
$$

where $f_a = \sum_{l=-\infty}^{+\infty}\left|\varphi_{a,l}\right|^2 f_0(\epsilon^F_a+l\Omega)$ is the Fermi Dirac staircase occupation from Eq. (3) in the main text. The above Greens functions can be used to generalize the calculations from the main text to compute un-equal time correlations. They can also be probed more directly in time-resolved ARPES measurements (see e.g., [47, 48]).