SUPPLEMENTAL MATERIAL FOR "THE ULTRA-CRITICAL FLOQUET NON-FERMI LIQUID"

Appendix A: Derivation of the electron collision integral

This section presents a detailed derivation of the electron collision integral, employing a formalism that combines fermionic and bosonic degrees of freedom from Ref.[13]. The full system contains fermionic and bosonic degrees of freedom. Fermionic creation and annihilation operators have the standard anticommutation relations:

$$\{\hat{a}_\alpha, \hat{a}^\dagger_\beta\} = \delta_{\alpha\beta}, \quad \{\hat{a}^\dagger_\alpha, \hat{a}^\dagger_\beta\} = 0, \quad \{\hat{a}_\alpha, \hat{a}_\beta\} = 0 \tag{A-1}$$

Here, α and β denote labels of fermionic states. The bosonic subsystem (i.e., the bath) is comprised of a collection of independent bosonic modes, labeled by q. Boson creation and annihilation operators follow the standard bosonic commutation relations:

$$[\hat{b}_q, \hat{b}^\dagger_{q'}] = \delta_{qq'}, \quad [\hat{b}^\dagger_q, \hat{b}^\dagger_{q'}] = 0, \quad [\hat{b}_q, \hat{b}_{q'}] = 0 \tag{A-2}$$

The analysis of Ref. [13], employs the concept of partial traces. The trace over the full many body Hilbert space can be split as $\mathrm{Tr} = \mathrm{Tr_e Tr_b}$, where $\mathrm{Tr_e}$ and $\mathrm{Tr_b}$ are partial traces over fermionic and bosonic degrees of freedom, respectively. We will further need to decompose the traces over bosonic degrees of freedom as follows: $\mathrm{Tr_b} = \mathrm{Tr_q Tr^q_b}$, $\mathrm{Tr^q_b} = \mathrm{Tr^{q'}_b Tr^{q,q'}_b}$, where $\mathrm{Tr^{q_1,q_2\cdots}_b}$ denotes the trace over bosonic modes excluding $(\mathrm{q_1, q_2 \ldots})$.

We begin by defining a one-electron density matrix (DM) $n^t_{\gamma\delta}$, which acts non-trivially as an operator on the bosonic degrees of freedom:

$$n^t_{\gamma\delta} \equiv \mathrm{Tr_e}[\eta_t \hat{a}^\dagger_\delta \hat{a}_\gamma] \tag{A-3}$$

The more standard one-body electronic density matrix $\rho_{\gamma\delta}$ can be obtained by tracing the above $n^t_{\gamma\delta}$ over bosonic degrees of freedom, namely:

$$\rho_{\gamma\delta} = \mathrm{Tr_b}[\hat{n}^t_{\gamma\delta}] \tag{A-4}$$

The equation of motion (EoM) for $n^t_{\gamma\delta}$ can be derived as follows (see also Ref. [13]):

$$\begin{aligned} i\hbar\partial_t \hat{n}^t_{\gamma\delta} &= \mathrm{Tr_e}\left\{[\hat{H}(t), \eta_t]\hat{a}^\dagger_\delta \hat{a}_\gamma\right\} \\ &= [\hat{H}_S(t), \hat{n}^t]_{\gamma\delta} + [\hat{H}_B, \hat{n}^t_{\gamma\delta}] + \sum_q [\chi_q \hat{b}_q + h.c., \hat{n}^t]_{\gamma\delta} + [\mathrm{Tr_e}(\hat{a}^\dagger_\nu \hat{a}^\dagger_\delta \hat{a}_\eta \hat{a}_\gamma \eta_t), \sum_q \hat{\chi}^{\nu\eta}_q \hat{b}_q + h.c.], \end{aligned} \tag{A-5}$$

where $\hat{H}(t)$ is defined in the main text. This leads to the EoM for the fermionic DM:

$$\hbar\partial_t \rho^t_{\gamma\delta} + i[\hat{H}_S(t), \hat{\rho}^t]_{\gamma\delta} = -i\mathrm{Tr_b}\Big(\sum_q [\hat{\chi}_q \hat{b}_q + h.c., n^t]_{\gamma\delta}\Big) = \mathcal{I}_{\gamma\delta}[t, \hat{\rho}(t)]. \tag{A-6}$$

We note that the term proportional to H_B vanishes due to the trace permutation property. Additionally, the term arising to the last term in Eq. (A-5) vanishes since b_q has no matrix structure in the fermionic subspace. We recognize the right-hand side of Eq. (A-6) as the collision integral. Our goal is to eliminate n^t from Eq. (A-6) using Eq. (A-5), in order to obtain a self-consistent equation that involves only the standard one-body electronic density matrix $\rho_{\gamma\delta}$.

To proceed with the elimination of n^t from Eq. (A-6), we begin by decomposing the trace over bosonic modes. We introduce the quantity $\hat{n}^{t,(q)} = \mathrm{Tr^q_b}[\hat{n}^t]$, which serves as a fundamental component for constructing the full bosonic trace in the collision integral from Eq. (A-6). The equation of motion for matrix elements of $\hat{n}^{t,(q)}$ can be derived as follows (see Ref. [13]):

$$\begin{aligned} i\hbar\partial_t \hat{n}^{t,(q)}_{\gamma\delta} &= [\hat{H}_S(t) + \hbar\omega_q \hat{b}^\dagger_q \hat{b}_q, \hat{n}^{t,(q)}]_{\gamma\delta} + [\hat{\chi}_q \hat{b}_q + h.c., \hat{n}^{t,(q)}]_{\gamma\delta} \\ &\quad + \sum_{q\neq q'} \mathrm{Tr^{q'}_b}[\hat{\chi}_{q'} \hat{b}_{q'} + h.c., \hat{n}^{t,(q,q')}]_{\gamma\delta} + \left[\mathrm{Tr^q_b Tr_e}(\hat{a}^\dagger_\nu \hat{a}^\dagger_\delta \hat{a}_\eta \hat{a}_\gamma \hat{\eta}_t), \hat{\chi}^{\nu\eta}_q \hat{b}_q + h.c.\right]. \end{aligned} \tag{A-7}$$

We now employ the assumption of weak coupling between fermions and the bosonic bath. As we can see from Eq. (A-6), to evaluate the collision integral up to the order $\hat{\chi}^2$, $\hat{n}^{t,(q)}$ has to be solved only in the linear order in $\hat{\chi}$. Thus, it is useful to split $\hat{n}^t$ into terms that are order zero in $\hat{\chi}$ and terms that vanish when $\hat{\chi} \to 0$, by introducing $\hat{\kappa}^{(q)}_{\gamma\delta} = \hat{n}^{(q)}_{\gamma\delta} - \hat{\bar{n}}^{(q)}_{\gamma\delta}$, where $\hat{\bar{n}}^{(q)}_{\gamma\delta}$ is the solution for $\hat{n}^t$ at zero coupling $\hat{\chi} = 0$. The scaling with $\hat{\chi}$ of $\hat{n}^{(q)}$ is expected to have the following structure:

$$\hat{n}^{(q)} = \begin{pmatrix} \hat{\rho}(t) & \sim \hat{\chi}_q \\ \sim \hat{\chi}^\dagger_q & \text{Thermal Bosons} \end{pmatrix} = \underbrace{\begin{pmatrix} \hat{\rho}(t) & 0 \\ 0 & \text{Thermal Bosons} \end{pmatrix}}_{\hat{\bar{n}}^{(q)}} + \underbrace{\begin{pmatrix} 0 & \sim \hat{\chi}_q \\ \sim \hat{\chi}^\dagger_q & 0 \end{pmatrix}}_{\hat{\kappa}_q} \tag{A-8}$$

It can be shown that the coupling-less component of $\hat{n}^t$ obeys the following EoMs:

$$\begin{aligned} i\hbar\partial_t \hat{\bar{n}}^{t,(q)}_{\gamma\delta} &= [\hat{H}_S(t) + \hbar\omega_q \hat{b}^\dagger_q \hat{b}_q, \hat{n}^{t,(q)}]_{\gamma\delta}, \\ i\hbar\partial_t \hat{\bar{n}}^{t,(q,q')}_{\gamma\delta} &= [\hat{H}_S(t) + \hbar\omega_q \hat{b}^\dagger_q \hat{b}_q + \hbar\omega_{q'} \hat{b}^\dagger_{q'} \hat{b}_{q'}, \hat{n}^{t,(q,q')}]_{\gamma\delta} \end{aligned} \tag{A-9}$$

Note that $\hat{n}^{t,(q,q')} = \mathrm{Tr}^{\mathrm{q,q'}}_{\mathrm{b}}[\hat{n}^t]$ has a diagonal structure in both q and q' modes, implying that the commutator proportional to $\sum_{\mathrm{q}\neq\mathrm{q'}}[\ldots]$ in Eq. (A-7) vanishes (is of higher order in $\hat{\chi}_q$). Thus we find the effective EoM for $n^{t,(q)}_{\gamma\delta}$ as:

$$\begin{aligned} i\hbar\partial_t \hat{n}^{t,(q)}_{\gamma\delta} &= [\hat{H}_S(t) + \hbar\omega_q \hat{b}^\dagger_q \hat{b}_q, \hat{n}^{t,(q)}]_{\gamma\delta} + [\hat{\chi}_q \hat{b}_q + h.c., \hat{n}^{t,(q)}]_{\gamma\delta} \\ &+ \left[\mathrm{Tr}^{\mathrm{q}}_{\mathrm{b}} \mathrm{Tr}_{\mathrm{e}}(\hat{a}^\dagger_\nu \hat{a}^\dagger_\delta \hat{a}_\eta \hat{a}_\gamma \hat{\eta}_t), \hat{\chi}^{\nu\eta}_q \hat{b}_q + h.c.\right] + \mathcal{O}(\hat{\chi}^2) \end{aligned} \tag{A-10}$$

This equation forms the basis for further analysis of the electron collision integral, taking into account the weak coupling approximation and the structure of the bosonic and fermionic subspaces.

1. Approximation of the four-fermion trace

This subsection presents an approximation for the four-fermion trace $\hat{g}^{\gamma\nu,t}_{\eta\delta,(q)} \equiv \mathrm{Tr}^{\mathrm{q}}_{\mathrm{b}} \mathrm{Tr}_{\mathrm{e}}(\hat{a}^\dagger_\nu \hat{a}^\dagger_\delta \hat{a}_\eta \hat{a}_\gamma \eta_t)$ to the zeroth order in $\hat{\chi}$. The fermionic commutation relations imply the following index permutation properties for $\hat{g}$:

$$\hat{g}^{\gamma\nu,t}_{\eta\delta,(q)} = \hat{g}^{\eta\delta,t}_{\gamma\nu,(q)} = -\hat{g}^{\eta\nu,t}_{\gamma\delta,(q)} = -\hat{g}^{\gamma\delta,t}_{\eta\nu,(q)}. \tag{A-11}$$

We employ an approximation proposed by Vasko and Raichev [13], which is similar to a time-dependent Hartree-Fock decomposition:

$$\hat{g}^{\gamma\nu,t}_{\eta\delta,(q)} \equiv \mathrm{Tr}^{\mathrm{q}}_{\mathrm{b}} \mathrm{Tr}_{\mathrm{e}}(\hat{a}^\dagger_\nu \hat{a}^\dagger_\delta \hat{a}_\eta \hat{a}_\gamma \hat{\eta}_t) \approx \frac{\hat{\bar{n}}^{t,(q)}_{\gamma\nu} \rho^t_{\eta\delta} - \hat{\bar{n}}^{t,(q)}_{\gamma\delta} \rho^t_{\eta\nu} + [(\gamma,\nu) \leftrightarrow (\eta,\delta)]}{2}, \tag{A-12}$$

This approximation is justified by the fact that both the exact quantity and the right-hand side of Eq. (A-12) obey the same equations of motion (EoMs) in the limit $\hat{\chi}_q \to 0$ [13]. To illustrate the self-consistency of this approximation, we compare the exact evolution of $\hat{g}$:

$$\begin{aligned} i\hbar\partial_t \hat{g}^{\gamma\nu,t}_{\eta\delta,(q)} &= \mathrm{Tr}^{\mathrm{q}}_{\mathrm{b}} \mathrm{Tr}_{\mathrm{e}}(\hat{a}^\dagger_\nu \hat{a}^\dagger_\delta \hat{a}_\eta \hat{a}_\gamma [\hat{H}(t), \hat{\eta}_t]) = \mathrm{Tr}^{\mathrm{q}}_{\mathrm{b}} \mathrm{Tr}_{\mathrm{e}}\big(\hat{a}^\dagger_\nu \hat{a}^\dagger_\delta \hat{a}_\eta \hat{a}_\gamma \big[\hat{H}_S(t) + \sum_{q'} \hbar\omega_{q'} \hat{b}^\dagger_{q'} \hat{b}_{q'}, \hat{\eta}_t\big]\big) \\ &= H^{\alpha\beta}_S(t)\big(\delta_{\gamma\alpha} \hat{g}^{\beta\nu,t}_{\eta\delta,(q)} - \delta_{\nu\beta} \hat{g}^{\gamma\alpha,t}_{\eta\delta,(q)} + \delta_{\eta\alpha} \hat{g}^{\gamma\nu,t}_{\beta\delta,(q)} - \delta_{\beta\delta} \hat{g}^{\gamma\nu,t}_{\eta\alpha,(q)}\big) + \hbar\omega_q \big[\hat{b}^\dagger_q \hat{b}_q, \hat{g}^{\gamma\nu,t}_{\eta\delta,(q)}\big]; \end{aligned} \tag{A-13}$$

with the approximated EoM:

$$\begin{aligned} i\hbar\partial_t \hat{g}^{\gamma\nu,t}_{\eta\delta,(q)} &= (i\hbar\partial_t \hat{\bar{n}}^{t,(q)}_{\gamma\nu})\rho^t_{\eta\delta} + \hat{\bar{n}}^{t,(q)}_{\gamma\nu}(i\hbar\partial_t \rho^t_{\eta\delta}) - (i\hbar\partial_t \hat{\bar{n}}^{t,(q)}_{\gamma\delta})\rho^t_{\eta\nu} - \hat{\bar{n}}^{t,(q)}_{\gamma\delta}(i\hbar\partial_t \rho_{\eta\nu}) + [(\gamma,\nu) \leftrightarrow (\eta,\delta)] \\ &= [\hat{H}_S(t), \hat{\bar{n}}^{t,(q)}]_{\gamma\nu} \rho^t_{\eta\delta} - [\hat{H}_S(t), \hat{\bar{n}}^{t,(q)}]_{\gamma\delta} \rho^t_{\eta\nu} + \hat{\bar{n}}^{t,(q)}_{\gamma\nu} [\hat{H}_S(t), \hat{\rho}^t]_{\eta\delta} \\ &\quad - \hat{\bar{n}}^{t,(q)}_{\gamma\delta} [\hat{H}_S(t), \hat{\rho}^t]_{\eta\nu} + \hbar\omega_q [\hat{b}^\dagger_q \hat{b}_q, \hat{g}^{\gamma\nu,t}_{\eta\delta,(q)}] + [(\gamma,\nu) \leftrightarrow (\eta,\delta)] \\ &= H^{\gamma\alpha}_S(t) \hat{g}^{\alpha\nu,t}_{\eta\delta,(q)} - H^{\alpha\delta}_S(t) \hat{g}^{\gamma\nu,t}_{\eta\alpha,(q)} + \hat{g}^{\gamma\nu,t}_{\alpha\delta,(q)} H^{\eta\alpha}_S(t) - \hat{g}^{\gamma\alpha,t}_{\eta\delta,(q)} H^{\alpha\nu}_S(t) + \hbar\omega_q [\hat{b}^\dagger_q \hat{b}_q, \hat{g}^{\gamma\nu,t}_{\eta\delta,(q)}] \\ &= H^{\alpha\beta}_S(t)\big(\delta_{\gamma\alpha} \hat{g}^{\beta\nu,t}_{\eta\delta,(q)} - \delta_{\beta\delta} \hat{g}^{\gamma\nu,t}_{\eta\alpha,(q)} + \delta_{\alpha\eta} \hat{g}^{\gamma\nu,t}_{\beta\delta,(q)} - \delta_{\beta\nu} \hat{g}^{\gamma\alpha,t}_{\eta\delta,(q)}\big) + \hbar\omega_q \big[\hat{b}^\dagger_q \hat{b}_q, \hat{g}^{\gamma\nu,t}_{\eta\delta,(q)}\big], \end{aligned} \tag{A-14}$$

where we have utilized the EoMs from Eqs. (A-9) and (A-6) in the limit $\hat{\chi}_q \to 0$. The evolution of the approximation is consistent with the exact evolution in Eq. (A-13), validating the approximation's applicability in this context.

2. Continuing the collision integral derivation

This subsection starts with the evaluation of $\hat{\kappa}$, which is essential for computing the collision integral to the desired order. Utilizing Eqs. (A-7), (A-9) and the definition of $\hat{\kappa}$ [see text above Eq. (A-8)], we derive the equation governing the evolution of $\hat{\kappa}$:

$$i\hbar\partial_t\hat{\kappa}^{t,(q)}_{\gamma\delta} = [\hat{H}_S(t) + \hbar\omega_q\hat{b}^\dagger_q\hat{b}_q, \hat{\kappa}^{t,(q)}]_{\gamma\delta} + [\hat{\chi}_q\hat{b}_q + h.c., \hat{n}^{t,(q)}]_{\gamma\delta} + \left[\hat{g}^{\gamma\nu,t}_{\eta\delta,(q)}, \hat{\chi}^{\nu\eta}_q\hat{b}_q + h.c.\right] \tag{A-15}$$

Here, g is a function of the equilibrium distribution $\bar{n}$ and electron density matrix ρ as defined in Eq. (A-12). The evolution of ρ, as shown in Eq. (A-6), can be simplified to:

$$i\hbar\partial_t\rho^t_{\gamma\delta} = [\hat{H}_S(t), \hat{\rho}^t]_{\gamma\delta} + \sum_q \mathrm{Tr_q}[\hat{\chi}_q\hat{b}_q + h.c., \hat{\kappa}^{t,(q)}]_{\gamma\delta} \tag{A-16}$$

This simplification arises from the fact that the bosonic equilibrium occupation function $\hat{\bar{n}} = N_b(b^\dagger b)$ is quadratic in b, causing the corresponding trace term to vanish.

We now arrive at a complete set of equations sufficient to determine the collision integral:

$$i\hbar\partial_t\hat{\bar{n}}^{t,(q)}_{\gamma\delta} = [\hat{H}_S(t) + \hbar\omega_q\hat{b}^\dagger_q\hat{b}_q, \hat{\bar{n}}^{t,(q)}]_{\gamma\delta} + \mathcal{O}(\hat{\chi}). \tag{A-17}$$

$$i\hbar\partial_t\hat{\kappa}^{t,(q)}_{\gamma\delta} = [\hat{H}_S(t) + \hbar\omega_q\hat{b}^\dagger_q\hat{b}_q, \hat{\kappa}^{t,(q)}]_{\gamma\delta} + [\hat{\chi}_q\hat{b}_q + h.c., \hat{\bar{n}}^{t,(q)}]_{\gamma\delta} + \left[\hat{g}^{\gamma\nu,t}_{\eta\delta,(q)}, \hat{\chi}^{\nu\eta}_q\hat{b}_q + h.c.\right] + \mathcal{O}(\hat{\chi}^2), \tag{A-18}$$

$$\hbar\partial_t\hat{\rho}^t_{\gamma\delta} + i[\hat{H}_S(t), \hat{\rho}^t]_{\gamma\delta} = -i\sum_q \mathrm{Tr_q}[\hat{\chi}_q\hat{b}_q + h.c., \hat{\kappa}^{t,(q)}]_{\gamma\delta} + \mathcal{O}(\hat{\chi}^3) = \mathcal{I}_{\gamma\delta}[t, \hat{\rho}(t)]. \tag{A-19}$$

By employing the electronic evolution operator [see Eq. (3) of the main text] and transitioning to the interaction picture using the unitary transformation $\hat{\mathcal{U}}_{t,t_0,(q)} = \hat{U}_S(t,t_0)e^{-i\omega_q\hat{b}^\dagger_q\hat{b}_q(t-t_0)}$, we can solve Eq. (A-18) to obtain:

$$\hat{\kappa}^{t,(q)}_{\alpha\beta} = \frac{1}{i\hbar}\int_{t_0}^{t} dt' \left(\hat{\mathcal{U}}^{\dagger,\alpha\gamma}_{t',t,(q)}[\hat{\chi}_q\hat{b}_q + h.c., \hat{\bar{n}}^{t',(q)}]_{\gamma\delta}\hat{\mathcal{U}}^{\delta\beta}_{t',t,(q)} + \hat{\mathcal{U}}^{\dagger,\alpha\gamma}_{t',t,(q)}\left[\hat{g}^{\gamma\nu,t'}_{\eta\delta,(q)}, \hat{\chi}^{\nu\eta}_q\hat{b}_q + h.c.\right]\hat{\mathcal{U}}^{\delta\beta}_{t',t,(q)}\right) \tag{A-20}$$

where $\hat{\bar{n}}^{t,(q)} = \hat{\mathcal{U}}_{t,t_0,(q)}\hat{\bar{n}}^{0,(q)}\hat{\mathcal{U}}^\dagger_{t,t_0,(q)}$, which is the solution of Eq. (A-17). Eq. (A-20) defines κ as a function of ρ.

The final step in deriving the kinetic equation is to substitute Eq. (A-20) into Eq. (A-19) and compute the partial traces of the collision integral:

$$\mathcal{I}_{\gamma\delta}[t, \hat{\rho}(t)] = -i\sum_q \mathrm{Tr_q}[\chi_q b_q + \chi^\dagger_q b^\dagger_q, \kappa^{t,(q)}]_{\gamma\delta} = -i\sum_q \left([\chi_q, \mathrm{Tr_q}\{b_q\kappa^{t,(q)}\} + [\chi^\dagger_q, \mathrm{Tr_q}\{b^\dagger_q\kappa^{t,(q)}\}]\right)_{\gamma\delta}, \tag{A-21}$$

and we note that $\mathcal{I}_{\gamma\delta}[t, \hat{\rho}(t)] = \mathcal{I}^{\mathrm{VR}}_{\gamma\delta}[t, \hat{\rho}(t)] + \mathcal{O}(\hat{\chi}^3)$. In the subsequent steps, we utilize the following relations:

$$\hat{\bar{n}}^{t,(q)} = \rho^t \otimes \bar{n}^{t,(q)}_b, \qquad \bar{n}^{t,(q)}_b = \frac{1}{Z_q}e^{-\beta\omega_q b^\dagger_q b_q}, \tag{A-22}$$

according to the definition from Eq. (A-8), and using the fact that $\mathrm{Tr_q} \equiv \sum_{n_q=0}^{\infty}\langle n_q| \dots |n_q\rangle$, we then compute the summation over n_q (not q) using the following identities:

$$\sum_{n_q=0}^{\infty} n_q\frac{e^{-\beta\omega_q n_q}}{Z_q} = \frac{1}{e^{\beta\omega_q}-1} = N_q, \qquad \sum_{n_q=0}^{\infty}(n_q+1)\frac{e^{-\beta\omega_q n_q}}{Z_q} = \frac{e^{\beta\omega_q}}{e^{\beta\omega_q}-1} = N_q + 1, \tag{A-23}$$

where $Z_q = e^{\beta\omega_q}/(e^{\beta\omega_q}-1)$ and $(N_q+1)e^{-\beta\omega_q} = N_q$. After performing all these summations, we derive the more general version of the collision integral as $\mathcal{I}^{\mathrm{VR}}[t, \hat{\rho}(t)] = I_e[t, \hat{\rho}(t)] + I_a[t, \hat{\rho}(t)]$, where the emission component is:

$$I_e[t, \hat{\rho}(t)] = \sum_q \int_{t_0}^{t} dt'(N_q+1)\Bigg(e^{-i\omega_q(t-t')}\left[U_S(t,t')[(1-\rho^{t'})\chi^\dagger_q\rho^{t'} + \rho^{t'}\mathrm{Tr}\{\rho^{t'}\chi^\dagger_q\}]U_S(t',t), \chi_q\right] - \\ e^{i\omega_q(t-t')}\left[U_S(t,t')[\rho^{t'}\chi_q(1-\rho^{t'}) + \rho^{t'}\mathrm{Tr}\{\rho^{t'}\chi_q\}]U_S(t',t), \chi^\dagger_q\right]\Bigg), \tag{A-24}$$

and the absorption component is:

$$I_a[t,\hat{\rho}(t)] = \sum_q \int_{t_0}^{t} dt' N_q \Bigg(e^{i\omega_q(t-t')} \left[U_S(t,t')[(1-\rho^{t'})\chi_q\rho^{t'} + \rho^{t'}\mathrm{Tr}\{\rho^{t'}\chi_q\}]U_S(t',t), \chi_q^\dagger \right] - e^{-i\omega_q(t-t')} \left[U_S(t,t')[\rho^{t'}\chi_q^\dagger(1-\rho^{t'}) + \rho^{t'}\mathrm{Tr}\{\rho^{t'}\chi_q^\dagger\}]U_S(t',t), \chi_q \right] \Bigg). \tag{A-25}$$

These equations provide a comprehensive description of the electron-boson collision integral and are identical to those derived by Vasko and Raichev in Ref.[13].

If we further impose translation invariance, i.e.,

$$\langle k|\rho|k'\rangle = \delta_{k,k'} f_k, \quad \langle k|\chi_q|k'\rangle \propto \delta_{k,k'+k_\lambda}, \tag{A-26}$$

where the label for boson modes are $q = (\mathbf{r}, \lambda)$ with λ labels the different boson modes coupled to each site, k_λ is the momentum for λ mode, f_k represents the occupation number for momentum k, the trace $\mathrm{Tr}\{\rho\chi_q\}$ is proportional to the identity matrix in momentum space that couples to the global particle number N:

$$\mathrm{Tr}\{\rho\chi_q\} = \sum_k \langle k|\rho\chi_q|k\rangle = \sum_k f_k \langle k|\chi_q|k\rangle \propto \delta_{0,k_\lambda} \sum_k f_k = \delta_{0,k_\lambda} N, \tag{A-27}$$

The collision integrals contain commutators of the following form vanishes:

$$\left[U_S(t,t')\rho^{t'}\mathrm{Tr}\{\rho^{t'}\chi_q\}U_S(t',t), \chi_q^\dagger \right] = \begin{cases} 0, & \text{when } k_\lambda \neq 0 \text{ since } \mathrm{Tr}\{\rho\chi_q\} = 0 \\ 0, & \text{when } k_\lambda = 0 \text{ since both } \mathrm{Tr}\{\rho\chi_q\} \text{ and } \chi_q \propto \text{ identity matrix} \end{cases} \tag{A-28}$$

Therefore, all terms proportional to $\mathrm{Tr}\{\rho\chi_q\}$ in the collision integrals in Eqs. (A-24) and (A-25) can be neglected. This allows us reduce the collision integrals to:

$$I_e[t,\hat{\rho}(t)] = \sum_q \int_{t_0}^{t} dt'(N_q+1)e^{-i\omega_q(t-t')} \left[U_S(t,t')[(1-\rho^{t'})\chi_q^\dagger\rho^{t'}]U_S(t',t), \chi_q \right] + h.c., \tag{A-29}$$

$$I_a[t,\hat{\rho}(t)] = \sum_q \int_{t_0}^{t} dt' N_q e^{i\omega_q(t-t')} \left[U_S(t,t')[(1-\rho^{t'})\chi_q\rho^{t'}]U_S(t',t), \chi_q^\dagger \right] + h.c. \tag{A-30}$$

In the main text we only display the emission component of the collision integral, as the absorption component can be obtained from the emission component by replacing $N_q + 1 \to N_q, \omega_q \to -\omega_q, \chi \to \chi^\dagger$.

Appendix B: Electron-boson collision integral for Floquet systems

This section applies the kinetic equation derived for Floquet systems. We adopt the following convention for the Floquet expansion and Fourier transformations:

$$f(\omega) = \int dt f(t)e^{i\omega t}, \quad f(t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} f(\omega), \quad f(t) = f(t+T) = \sum_n f_n e^{-in\omega t}. \tag{B-1}$$

This allows us to express the evolution operator using the Floquet theorem:

$$i\hbar\partial_t \hat{U}_S(t,t_0) = \hat{H}_S(t)\hat{U}_S(t,t_0), \quad U_S(t,t') = \sum_k |\psi_k(t)\rangle\langle\psi_k(t')|, \quad |\psi_k(t)\rangle = \sum_l e^{-i(\epsilon_k^F + l\Omega)t} |\varphi_{k,l}\rangle, \tag{B-2}$$

where $\Omega = 2\pi/T$ is the frequency of the drive $H_S(t) = H_S(t+T)$ and ϵ_k^F is the system's Floquet energy. Now, let us further specialize to case of the single-band system:

$$U_S(t,t') = \sum_k \sum_{ll'} e^{-i\epsilon_k^F(t-t') - il\Omega t + il'\Omega t'} \varphi_{k,l}\varphi_{k,l'}^* |k\rangle\langle k|, \quad \rho^t = \sum_k f_{k,t} |k\rangle\langle k|, \tag{B-3}$$

the reduced single-band Floquet-Boltzmann equation can be obtained by computing $\partial_t f_{k,t} = \langle k| J_e[t,\rho]|k\rangle + \langle k| J_a[t,\rho]|k\rangle$. Using Eqs. (A-29), (B-3), we derive:

$$\begin{aligned}\langle k| I_e[t,\rho]|k\rangle = &\sum_{l_1 l_2 l_3 l_4}\sum_{k',q}\int_{t_0}^{t} dt'(N_q+1)e^{-i\omega_q(t-t')-il_1\Omega t+il_2\Omega t'}e^{-il_3\Omega t'+il_4\Omega t}\\ &\times\Big[\varphi_{k,l_1}\varphi^*_{k_1,l_2}\varphi_{k',l_3}\varphi^*_{k',l_4}e^{-i(\epsilon^F_k-\epsilon^F_{k'})(t-t')}|\langle k'|\chi_q|k\rangle|^2(1-f_{k,t'})f_{k',t'}\\ &-\varphi_{k',l_1}\varphi^*_{k',l_2}\varphi_{k,l_3}\varphi^*_{k,l_4}e^{i(\epsilon^F_k-\epsilon^F_{k'})(t-t')}|\langle k|\chi_q|k'\rangle|^2(1-f_{k',t'})f_{k,t'}\Big]+h.c.\end{aligned} \tag{B-4}$$

The time integration can be performed assuming that the interaction with the bath started in the distant past, and the system has reached a steady state with respect to its interaction with the bath, without retaining its initial information. This allows us to extend the lower limit of integration to negative infinity $\int_{t_0}^{t} dt' \to \int_{-\infty}^{t} dt' e^{\eta t'}$ and express $f_{k,t'}$ into Floquet expansion:

$$\begin{aligned}\langle k| I_e[t,\rho]|k\rangle = &\sum_{l_1\ldots l_6}\sum_{k',q}(N_q+1)e^{-i(l_1-l_2+l_3-l_4+l_5+l_6)\Omega t}\\ \times&\left[\frac{\varphi_{k,l_1}\varphi^*_{k,l_2}\varphi_{k',l_3}\varphi^*_{k',l_4}|\langle k'|\chi_q|k\rangle|^2\bar{f}_{kl_5}f_{k'l_6}}{i(\omega_q+(l_2-l_3-l_5-l_6)\Omega+\epsilon^F_k-\epsilon^F_{k'})+\eta}-\frac{\varphi_{k',l_1}\varphi^*_{k',l_2}\varphi_{k,l_3}\varphi^*_{k,l_4}|\langle k|\chi_q|k'\rangle|^2\bar{f}_{k'l_5}f_{kl_6}(t')}{i(\omega_q-\epsilon^F_k+\epsilon^F_{k'}+(l_2-l_3-l_5-l_6)\Omega)+\eta}\right]+h.c.,\end{aligned} \tag{B-5}$$

where we used the compact notation $\bar{f}_{k,t} = 1 - f_{k,t}$, with the Floquet modes $\bar{f}_{l,k} = \delta_{l0} - f_{l,k}$. By projecting the Boltzmann equation onto Floquet mode l, we can compactly write the complete Floquet-Boltzmann equation as:

$$\begin{aligned}-il\Omega f_{kl} = &\sum_{s=e,a}\sum_{l_1\ldots l_6}\sum_{k',q}N^s_q\delta^{l-l_5-l_6}_{l_1-l_2+l_3-l_4}\left[\frac{\varphi_{k,l_1}\varphi^*_{k,l_2}\varphi_{k',l_3}\varphi^*_{k',l_4}|\langle k'|\chi^s_q|k\rangle|^2\bar{f}_{kl_5}f_{k'l_6}}{i(\mathrm{sgn_s}\omega_q+(l_2-l_3-l_5-l_6)\Omega+\epsilon^F_k-\epsilon^F_{k'})+\eta}-(k\leftrightarrow k')\right]+\\ &\sum_{s=e,a}\sum_{l_1\ldots l_6}\sum_{k',q}N^s_q\delta^{l-l_5-l_6}_{l_1-l_2+l_3-l_4}\left[\frac{\varphi_{k,l_1}\varphi^*_{k,l_2}\varphi_{k',l_3}\varphi^*_{k',l_4}|\langle k'|\chi^s_q|k\rangle|^2\bar{f}_{kl_5}f_{k'l_6}}{-i(\mathrm{sgn_s}\omega_q+(l_2-l_3+l)\Omega+\epsilon^F_k-\epsilon^F_{k'})+\eta}-(k\leftrightarrow k')\right].\end{aligned} \tag{B-6}$$

Here $N^e_q = N_q + 1, N^a_q = N_q$, the sign in the denominator $\mathrm{sgn_e} = 1, \mathrm{sgn_a} = -1$ and $\delta^b_a \equiv \delta_{ab}$ is the Kronecker delta. Importantly, the right-hand side of Eq. (B-6) is of order $\hat{\chi}^2$. This implies that in the limit of zero coupling to the bath $\hat{\chi} \to 0$, all oscillating components ($l \neq 0$) of $f_{k,l}$ vanish, and the system's state is defined by the self-consistent solution of the Floquet-Boltzmann equation for the mode $l = 0$. Consequently, the solution of the driven Boltzmann equation with weak coupling to the bath is time-independent. The corresponding static emission component of the collision integral is (here $f_{k,0} \equiv f_k$):

$$\begin{aligned}\langle k| I_e[\rho]|k\rangle = &\sum_{k'}\sum_{l_1 l_2 l_3 l_4}\sum_{q}(N_q+1)\delta_{l_1+l_3,l_2+l_4}\\ \times&\left[\frac{\varphi_{k,l_1}\varphi^*_{k,l_2}\varphi_{k',l_3}\varphi^*_{k',l_4}|\langle k'|\chi_q|k\rangle|^2(1-f_k)f_{k'}}{i(\omega_q+\epsilon^F_k-\epsilon^F_{k'}+l_2\Omega-l_3\Omega)+\eta}-\frac{\varphi_{k',l_1}\varphi^*_{k',l_2}\varphi_{k,l_3}\varphi^*_{k,l_4}|\langle k|\chi_q|k'\rangle|^2(1-f_{k'})f_k}{i(\omega_q-\epsilon^F_k+\epsilon^F_{k'}+l_2\Omega-l_3\Omega)+\eta}\right]+h.c.\end{aligned} \tag{B-7}$$

Focusing on the Hermitian conjugate term, after relabelling $(l_1, l_3 \leftrightarrow l_2, l_4)$ and using the Kronecker delta condition $l_4 - l_1 = l_3 - l_2$, we observe that denominators combine into delta functions as $\frac{1}{ix+\eta} + \frac{1}{-ix+\eta} = \frac{2\eta}{x^2+\eta^2} \to 2\pi\delta(x)$. Thus, by introducing the scattering matrix elements:

$$W^e_{k\to k_1} = 2\pi\sum_{l_1 l_2 l_3 l_4}\sum_{q}(N_q+1)\delta_{l_1+l_3,l_2+l_4}\delta(\omega_q+\epsilon^F_{k_1}-\epsilon^F_k+l_2\Omega-l_3\Omega)\mathrm{Re}[\varphi_{k_1,l_1}\varphi^*_{k_1,l_2}\varphi_{k,l_3}\varphi^*_{k,l_4}]|\langle k|\chi_q|k_1\rangle|^2. \tag{B-8}$$

the collision integral in Eq. (B-7), together with the absorption component, can be written as:

$$\langle k|\mathcal{I}[\rho]|k\rangle = \sum_{s=e,a}\sum_{k'}[W^s_{k'\to k}f_{k'}(1-f_k) - W^s_{k\to k'}f_k(1-f_{k'})], \tag{B-9}$$

which provides the Floquet-Boltzmann equation for a single-band Floquet system.

Appendix C: Analytical analysis of non-analyticities

This section presents derivations for the scattering matrix and formal expressions for the steady-state occupation and its derivatives. Based on these derivations, we demonstrate the prevalence of non-analyticities in the steady-state occupation.

1. Scattering matrix element $W(q \to p)$

We begin with the expression for the scattering matrix $W_e(q \to p)$ [see Eq. (B-8)]:

$$W_e(q \to p) = 2\pi \sum_{l_1 l_2 l_3 l_4} \mathrm{Re}[\varphi_{q,l_1}\varphi^*_{q,l_2}\varphi_{p,l_3}\varphi^*_{p,l_4}]\delta_{l_1+l_3,l_2+l_4} \sum_\lambda [N_b(\lambda)+1]\delta(\omega_\lambda + \epsilon^F_q - \epsilon^F_p + l_2\Omega - l_3\Omega)|\langle q|\,\chi_\lambda\,|p\rangle|^2$$
$$= 2\pi \sum_l \sum_{l_1} |\varphi_{l_1,p}\varphi^*_{l_1+l,q}|^2 \sum_\lambda [N_b(\lambda)+1]\delta(\omega_\lambda + \epsilon^F_q - \epsilon^F_p + l\Omega)|\langle q|\,\chi_\lambda\,|p\rangle|^2, \tag{C-1}$$

in which we let $l = l_2 - l_3$. We introduce the density of states $\nu_B(\epsilon)$ for the bosonic bath:

$$\nu_B(\epsilon) = \sum_\lambda \delta(\epsilon - \omega_\lambda). \tag{C-2}$$

For an arbitrary function $f(\omega_\lambda)$, we can use the following property:

$$\sum_\lambda f(\omega_\lambda)\delta(\omega_\lambda - x) = f(x)\sum_\lambda \delta(\omega_\lambda - x) = f(x)\nu_B(x). \tag{C-3}$$

This property can be verified by multiplying both sides by any test function $g(x)$ and integrating over x. Using Eqs. (C-2) and (C-3), we can reformulate the sum over λ in terms of ν_B in Eq. (C-1):

$$W_e(q \to p) = \sum_{l\in\mathbb{Z}} \Gamma_l(q,p) \times \left[N_b(\epsilon^F_q - \epsilon^F_p + l\Omega) + 1\right] \times \nu_B(\epsilon^F_q - \epsilon^F_p + l\Omega), \tag{C-4}$$
$$W_a(q \to p) = \sum_{l\in\mathbb{Z}} \Gamma_l(q,p) \times N_b(\epsilon^F_p - \epsilon^F_q - l\Omega) \times \nu_B(\epsilon^F_p - \epsilon^F_q - l\Omega). \tag{C-5}$$

Here, $\Gamma_l(q,p)$ is defined as:

$$\Gamma_l(q,p) \equiv 2\pi|M_l(q,p)|^2 \sum_{l_1\in\mathbb{Z}} |\varphi_{l_1,p}\varphi^*_{l_1+l,q}|^2 = 2\pi|M_l(q,p)|^2\Phi^{(l)}_{q,p} = \Gamma_l(p,q), \tag{C-6}$$
$$|M_l(q,p)|^2 = |\langle q|\,\chi(|\epsilon^F_q - \epsilon^F_p + l\Omega|)\,|p\rangle|^2. \tag{C-7}$$

We note that $W_a(q \to p)$ is obtained from $W_e(q \to p)$ by replacing $N_b(\lambda)+1 \to N_b(\lambda)$, $\omega_\lambda \to -\omega_\lambda$, and $\chi_\lambda \to \chi^\dagger_\lambda$. Using the symmetry $\Gamma_l(q,p) = \Gamma_l(p,q)$ and observing the pattern of appearances of $\epsilon^F_p - \epsilon^F_q - l\Omega$ in $W_e(q \to p)$ and $W_a(q \to p)$, we obtain the total scattering matrix element:

$$W(q \to p) = W_e(q \to p) + W_a(q \to p) = \sum_{l\in\mathbb{Z}} \Gamma_l(q,p) S(\varepsilon_q - \varepsilon_p + l\Omega), \tag{C-8}$$

where we define $\varepsilon_p \equiv \epsilon^F_q$ for conciseness, and the function $S(x)$ as:

$$S(x) \equiv [N_b(|x|) + \Theta(x)]\,\nu_B(|x|). \tag{C-9}$$

In the main text, we employ a simplified bath consisting of dispersionless "Einstein" phonons for clarity. While this simplification proves instructive, our general formalism for the scattering matrix element given by Eq. (C-8) remains applicable to more realistic bosonic baths. For instance, when considering a physical phonon bath, one can incorporate the appropriate boson dispersion relation and modify the fermion-boson coupling matrix $\hat{\chi}_\lambda$ accordingly.

2. Formal expressions for the steady state occupation and its derivatives

In this subsection, we show formal expressions for the steady-state occupation and its derivatives. We start from the steady-state condition [Eq. (5) in the main text]:

$$\sum_q (f_q W_{q\to p}\bar{f}_p - f_p W_{p\to q}\bar{f}_q) = 0, \quad \bar{f}_p \equiv 1 - f_p, \tag{C-10}$$

where f_p is the occupation at momentum p, $W_{q\to p}$ is the scattering rate from momentum q to p, and $\bar{f}_p$ is the vacancy at momentum p.

We relabel the momentum p by $\{\varepsilon_p, \eta_p\}$, where ε_p is the quasi-energy and η_p is a generic parameter, which parametrizes the equi-energy surface at ε_p [e.g., (θ, ϕ) angles in 3D]. This yields:

$$\int_{\varepsilon_b}^{\varepsilon_t} d\varepsilon_q \oint d\eta_q |J_{\varepsilon_q,\eta_q}| (f_{\varepsilon_q,\eta_q} W_{\varepsilon_q,\eta_q\to\varepsilon_p,\eta_p} \bar{f}_{\varepsilon_p,\eta_p} - f_{\varepsilon_p,\eta_p} W_{\varepsilon_p,\eta_p\to\varepsilon_q,\eta_q} \bar{f}_{\varepsilon_q,\eta_q}) = 0, \tag{C-11}$$

where J_{ε_q,η_q} is the Jacobian for the transformation $p \to \{\varepsilon_p, \eta_p\}$.

We then take the n-th derivative with respect to ε_p on both sides, apply the general Leibniz rule, and obtain:

$$f^{[n]}_{\varepsilon_p,\eta_p} = \frac{1}{R_{\varepsilon_p,\eta_p}} \int_{\varepsilon_b}^{\varepsilon_t} d\varepsilon_q \oint d\eta_q |J_{\varepsilon_q,\eta_q}| \sum_{k=0}^{n-1} \binom{n}{k} \Big(f_{\varepsilon_q,\eta_q} W^{[n-k]}_{\varepsilon_q,\eta_q\to\varepsilon_p,\eta_p} \bar{f}^{[k]}_{\varepsilon_p,\eta_p} - f^{[k]}_{\varepsilon_p,\eta_p} W^{[n-k]}_{\varepsilon_p,\eta_p\to\varepsilon_q,\eta_q} \bar{f}_{\varepsilon_q,\eta_q} \Big), \quad (n \geq 1) \tag{C-12}$$

where the k-th derivative of a function g with respect to ϵ_p is denoted by

$$g^{[k]}(\varepsilon_p) \equiv \frac{d^k g(\varepsilon_p)}{d\varepsilon_p^k}, \tag{C-13}$$

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the binomial coefficient, and R_{ε_p,η_p} represents the maximally allowed scattering rate at momentum p:

$$R_{\varepsilon_p,\eta_p} = \int_{\varepsilon_b}^{\varepsilon_t} d\varepsilon_q \oint d\eta_q |J_{\varepsilon_q,\eta_q}| (f_{\varepsilon_q,\eta_q} W_{\varepsilon_q,\eta_q\to\varepsilon_p,\eta_p} + W_{\varepsilon_p,\eta_p\to\varepsilon_q,\eta_q} \bar{f}_{\varepsilon_q,\eta_q}) = \sum_q \big(f_q W_{q\to p} + W_{p\to q}\bar{f}_q \big). \tag{C-14}$$

We further apply the general Leibniz rule on $W^{[n-k]}_{\varepsilon_q,\eta_q\to\varepsilon_p,\eta_p}$ and $W^{[n-k]}_{\varepsilon_p,\eta_p\to\varepsilon_q,\eta_q}$ [see Eq. (C-8)]:

$$\begin{aligned} W^{[n-k]}_{\varepsilon_q,\eta_q\to\varepsilon_p,\eta_p} &= \sum_{l\in\mathbb{Z}} \sum_{j=0}^{n-k} \binom{n-k}{j} \Big[\Gamma_l(\varepsilon_q,\eta_q;\varepsilon_p,\eta_p) \Big]^{[n-k-j]} S^{[j]}(\varepsilon_q - \varepsilon_p + l\Omega), \\ W^{[n-k]}_{\varepsilon_p,\eta_p\to\varepsilon_q,\eta_q} &= \sum_{l\in\mathbb{Z}} \sum_{j=0}^{n-k} \binom{n-k}{j} \Big[\Gamma_l(\varepsilon_q,\eta_q;\varepsilon_p,\eta_p) \Big]^{[n-k-j]} S^{[j]}(\varepsilon_p - \varepsilon_p + l\Omega). \end{aligned} \tag{C-15}$$

Combining these results, we obtain:

$$\begin{aligned} R_{\varepsilon_p,\eta_p} \cdot f^{[n]}_{\varepsilon_p,\eta_p} = \sum_{l\in\mathbb{Z}} \sum_{k=0}^{n-1} \sum_{j=0}^{n-k} \binom{n}{k} \binom{n-k}{j} \int_{\varepsilon_b}^{\varepsilon_t} d\varepsilon_q \oint d\eta_q |J_{\varepsilon_q,\eta_q}| \Big[\Gamma_l(\varepsilon_q,\eta_q;\varepsilon_p,\eta_p) \Big]^{[n-k-j]} \\ \times \Big(f_{\varepsilon_q,\eta_q} S^{[j]}(\varepsilon_q - \varepsilon_p + l\Omega) \bar{f}^{[k]}_{\varepsilon_p,\eta_p} - \bar{f}_{\varepsilon_q,\eta_q} S^{[j]}(\varepsilon_p - \varepsilon_q + l\Omega) f^{[k]}_{\varepsilon_p,\eta_p} \Big). \end{aligned} \tag{C-16}$$

Finally, we can compactify the expression further and obtain:

$$f^{[n]}_{\varepsilon_p,\eta_p} = \sum_{l\in\mathbb{Z}} \sum_{k=0}^{n-1} \sum_{j=0}^{n-k} \int_{\varepsilon_b}^{\varepsilon_t} d\varepsilon_q \Big(B^{lnkj}_{\varepsilon_p,\eta_p}(\varepsilon_q) S^{[j]}(\varepsilon_q - \varepsilon_p + l\Omega) \bar{f}^{[k]}_{\varepsilon_p,\eta_p} - \bar{B}^{lnkj}_{\varepsilon_p,\eta_p}(\varepsilon_q) S^{[j]}(\varepsilon_p - \varepsilon_q + l\Omega) f^{[k]}_{\varepsilon_p,\eta_p} \Big), \tag{C-17}$$

where the auxiliary functions $B^{lnkj}_{\varepsilon_p,\eta_p}(\varepsilon_q)$ and $\bar{B}^{lnkj}_{\varepsilon_p,\eta_p}(\varepsilon_q)$ are defined as:

$$\begin{aligned} B^{lnkj}_{\varepsilon_p,\eta_p}(\varepsilon_q) &= \frac{1}{R_{\varepsilon_p,\eta_p}} \binom{n}{k} \binom{n-k}{j} \oint d\eta_q |J_{\varepsilon_q,\eta_q}| \big[\Gamma_l(\varepsilon_q,\eta_q;\varepsilon_p,\eta_p) \big]^{[n-k-j]} \times f_{\varepsilon_q,\eta_q}, \\ \bar{B}^{lnkj}_{\varepsilon_p,\eta_p}(\varepsilon_q) &= \frac{1}{R_{\varepsilon_p,\eta_p}} \binom{n}{k} \binom{n-k}{j} \oint d\eta_q |J_{\varepsilon_q,\eta_q}| \big[\Gamma_l(\varepsilon_q,\eta_q;\varepsilon_p,\eta_p) \big]^{[n-k-j]} \times \bar{f}_{\varepsilon_q,\eta_q}. \end{aligned} \tag{C-18}$$

The calculation of derivatives is more transparent in momentum space:

$$f_p^{\{n\}} = \sum_q \sum_{k=0}^{n-1} \binom{n}{k} \frac{f_q W_{q\to p}^{\{n-k\}} \bar{f}_p^{\{k\}} - f_p^{\{k\}} W_{p\to q}^{\{n-k\}} \bar{f}_q}{R_p}, \quad R_p = \sum_q \left(f_q W_{q\to p} + W_{p\to q} \bar{f}_q \right) \tag{C-19}$$

where the superscript $\{k\}$ on a function denotes its k-th derivative with respect to the momentum p. The momentum-space formulation of Eq. (C-19) offers a compact representation; however, it obscures the essential role played by S_x. In contrast, Eq. (C-17) underscores that non-analyticities arise exclusively from those present in S_x, which exhibits a sole dependence on the quasi-energy ε_p and remains independent of the orientation η_p characterizing the equi-energy surface.

3. Generic analysis of non-analyticities

This subsection provides explicit calculations supporting the generic analysis of non-analyticities discussed in the main text. Consider an integral of the form:

$$R(\varepsilon) = \int_{\varepsilon_b}^{\varepsilon_t} d\varepsilon' r(\varepsilon, \varepsilon') \delta(\varepsilon' - \varepsilon + \Lambda), \tag{C-20}$$

where $r(\varepsilon, \varepsilon')$ is a continuous function. Evaluating $R(\varepsilon)$ in the vicinity of $\varepsilon = \varepsilon_{b,t} + \Lambda$ yields:

$$\begin{aligned} R(\varepsilon = \varepsilon_b + \Lambda + 0^+) &= \int_{\varepsilon_b}^{\varepsilon_t} d\varepsilon' r(\varepsilon_b + \Lambda + 0^+, \varepsilon') \delta(\varepsilon' - \varepsilon_b - 0^+) = r(\varepsilon_b + \Lambda + 0^+, \varepsilon_b + 0^+), \\ R(\varepsilon = \varepsilon_b + \Lambda - 0^+) &= \int_{\varepsilon_b}^{\varepsilon_t} d\varepsilon' r(\varepsilon_b + \Lambda - 0^+, \varepsilon') \delta(\varepsilon' - \varepsilon_b + 0^+) = 0, \end{aligned} \tag{C-21}$$

and

$$\begin{aligned} R(\varepsilon = \varepsilon_t + \Lambda + 0^+) &= \int_{\varepsilon_b}^{\varepsilon_t} d\varepsilon' r(\varepsilon_t + \Lambda + 0^+, \varepsilon') \delta(\varepsilon' - \varepsilon_t - 0^+) = 0, \\ R(\varepsilon = \varepsilon_t + \Lambda - 0^+) &= \int_{\varepsilon_b}^{\varepsilon_t} d\varepsilon' r(\varepsilon_t + \Lambda - 0^+, \varepsilon') \delta(\varepsilon' - \varepsilon_t + 0^+) = r(\varepsilon_t + \Lambda - 0^+, \varepsilon_t - 0^+). \end{aligned} \tag{C-22}$$

Therefore non-analyticities manifest in $R(\varepsilon)$ near $\varepsilon = \varepsilon_{b,t} + \Lambda$ when the following condition is satisfied:

$$R(\varepsilon_{b,t} + \Lambda + 0^+) - R(\varepsilon_{b,t} + \Lambda - 0^+) = \pm r(\varepsilon_{b,t} + \Lambda, \varepsilon_{b,t}) \neq 0. \tag{C-23}$$

The analysis above, combined with the expression for the n-th derivative of the occupation function in Eq. (C-17), clarifies mathematically the origins of apparent non-analyticities or discontinuities in $f_{\varepsilon_p,\eta_p}^{[n]}$. Equation (C-17), repeated here for convenience,

$$f_{\varepsilon_p,\eta_p}^{[n]} = \sum_{l\in\mathbb{Z}} \sum_{k=0}^{n-1} \sum_{j=0}^{n-k} \int_{\varepsilon_b}^{\varepsilon_t} d\varepsilon_q \Big(B_{\varepsilon_p,\eta_p}^{lnkj}(\varepsilon_q) S^{[j]}(\varepsilon_q - \varepsilon_p + l\Omega) \bar{f}_{\varepsilon_p,\eta_p}^{[k]} - \bar{B}_{\varepsilon_p,\eta_p}^{lnkj}(\varepsilon_q) S^{[j]}(\varepsilon_p - \varepsilon_q + l\Omega) f_{\varepsilon_p,\eta_p}^{[k]} \Big),$$

has a structure similar to that of $R(\varepsilon)$ in Eq. (C-20). If either $S^{[j]}(\varepsilon_q - \varepsilon_p + l\Omega)$, the j-th derivative of S with respect to ε_p, or $f_{\varepsilon_p,\eta_p}^{[k]}$, the k-th derivative of f_{ε_p,η_p} with respect to ε_p, contains a Dirac delta function, it can lead to discontinuities in $f_{\varepsilon_p,\eta_p}^{[n]}$. This conclusion follows the same reasoning as the one used to explain the discontinuity in $R(\varepsilon)$ through Eqs. (C-20) to (C-23). Therefore, the presence of Dirac delta functions in the derivatives of S or f_{ε_p,η_p} is responsible for the non-analytic or discontinuous behavior of $f_{\varepsilon_p,\eta_p}^{[n]}$.

Appendix D: Analytical analysis of S function for different types of bath

In Section C, we demonstrated that the behavior of S function can lead to non-analyticities in the system's occupation. To support the analysis in the main text, we now perform a detailed analytical examination of the S function for the Ohmic and gapped baths.

1. Ohmic bath with a finite temperature

We first consider the following S function with a finite bath temperature:

$$S(\omega) = [N_b(|\omega|) + \Theta(\omega)]\nu_B(|\omega|), \quad \nu_B(\omega) = (c_1\omega + c_2\omega^2)\Theta(\omega), \tag{D-1}$$

where $N_b(\omega)$ is the Bose-Einstein distribution and $\Theta(\omega)$ is the Heaviside step function. More explicitly,

$$S(\omega) = c_1 y(\omega) + c_2 y(\omega)|x|, \quad y(\omega) = \frac{\omega e^{\beta\omega}}{e^{\beta\omega} - 1} \text{ is a smooth function}, \quad y(0) = 1/\beta. \tag{D-2}$$

By taking derivatives of $S(\omega)$ with respect to ω, we obtain

$$S^{[1]}(\omega) = c_1 y^{[1]}(\omega) + c_2 y^{[1]}(\omega)\,|\omega| + c_2 y(\omega)\,\mathrm{sgn}(\omega), \tag{D-3}$$

$$S^{[2]}(\omega) = c_1 y^{[2]}(\omega) + c_2 y^{[2]}(\omega)\,|\omega| + 2c_2 y^{[1]}(\omega)\,\mathrm{sgn}(\omega) + \frac{2c_2}{\beta}\,\delta(\omega), \tag{D-4}$$

during which we used the identity $y(x)\delta(x) = y(0)\delta(x)$. Consequently, we observe that the second-order derivative $S^{[2]}(\omega)$ is the lowest order derivative containing a Dirac delta function at finite bath temperature. According to Eq. (C-17) and following the reasoning used to explain the discontinuity in $R(\varepsilon)$ through Eqs. (C-20) to (C-23), the Dirac delta function in $S^{[2]}$ leads to discontinuities at $l\Omega$ in $f^{[2]}(\epsilon)$ of the system's occupation function at finite bath temperature.

As a concrete example, following Eq. (F-5) in the Appendix F, for parabolic bands in 1D and 2D, we have

$$f^{[2]}(l\Omega^+) - f^{[2]}(l\Omega^-) \propto \frac{2c_2}{\beta}\,\varrho^d(0^+)\Gamma_l^d(0^+, l\Omega^+)\big[f(0^+)\bar{f}(l\Omega^+) - \bar{f}(0^+)f(l\Omega^+)\big]. \tag{D-5}$$

In the above equation, $l\Omega^\pm \equiv l\Omega \pm 0^+$, where 0^+ is a positive infinitesimal, $\Gamma_l^d(\epsilon_q, \epsilon_p)$ is the scattering amplitude corresponding to the scattering matrix $W^d_{\epsilon_p \to \epsilon_q} = \sum_l \Gamma_l^d(\epsilon_q, \epsilon_p) S(\epsilon_q - \epsilon_p + l\Omega)$, and $\varrho^d(\epsilon) = \epsilon^{(d-2)/2}$ is the density of states, both for the d-dimensional ($d = 1, 2$) parabolic model. Since $\epsilon = k^2/2m$ for parabolic bands, the discontinuities in the above second energy derivative $f^{[2]}(\epsilon) = d^2 f(\epsilon)/d\epsilon^2$ can be mapped to discontinuities in $d^2 f(k)/dk^2$ through the chain rule of differentiation.

2. Ohmic bath with zero temperature

We now consider the following S function with zero bath temperature:

$$S(\omega) = \Theta(\omega)\nu_B(|\omega|) = c_1\omega\Theta(\omega) + c_2\omega^2\Theta(\omega), \quad \nu_B(\omega) = (c_1\omega + c_2\omega^2)\Theta(\omega), \tag{D-6}$$

where we take $N_b(\omega) = 0$ in this case. By taking derivatives of $S(\omega)$ with respect to ω, we obtain

$$S^{[1]}(\omega) = c_1\Theta(\omega) + 2c_2\omega\Theta(\omega) \tag{D-7}$$

$$S^{[2]}(\omega) = c_1\delta(\omega) + 2c_2\Theta(\omega) \tag{D-8}$$

during which we used the identity $x\delta(x) = 0$. We observe that the second-order derivative $S^{[2]}(\omega)$ is the lowest order derivative containing a Dirac delta function at zero temperature. According to Eq. (C-17) and following the reasoning used to explain the discontinuity in $R(\varepsilon)$ through Eqs. (C-20) to (C-23), this leads to discontinuities at $l\Omega$ in $f^{[2]}(\epsilon)$ of the system's occupation function at finite bath temperature.

As a concrete example, following Eq. (F-5) in the Appendix F, for parabolic bands in 1D and 2D, we have in this case

$$f^{[2]}(l\Omega^+) - f^{[2]}(l\Omega^-) \propto c_1\,\varrho^d(0^+)\Gamma_l^d(0^+, l\Omega^+)\big[f(0^+)\bar{f}(l\Omega^+) - \bar{f}(0^+)f(l\Omega^+)\big], \tag{D-9}$$

where $l\Omega^\pm \equiv l\Omega \pm 0^+$, $\Gamma_l^d(\epsilon_q, \epsilon_p)$ is the scattering amplitude and $\varrho^d(\epsilon) = \epsilon^{(d-2)/2}$ is the density of states, both for the d-dimensional ($d = 1, 2$) parabolic model. The discontinuities in the above second energy derivative $f^{[2]}(\epsilon) = d^2 f(\epsilon)/d\epsilon^2$ can be mapped to discontinuities in $d^2 f(k)/dk^2$ through the chain rule of differentiation.

3. Gapped bath with a finite temperature

Here we consider the following S function with a finite bath temperature:

$$S(\omega) = [N_b(|\omega|) + \Theta(\omega)]\nu_B(|\omega|), \quad \nu_B(\omega) = \Theta(\omega - \Delta), \quad \Delta > 0. \tag{D-10}$$

The first-order derivative of $S(\omega)$ with respect to ω already contains Dirac delta functions:

$$S^{[1]}(\omega) = \begin{cases} \partial_\omega N_b(\omega)\,\Theta(+\omega - \Delta) + [N_b(|\omega|) + 1]\delta(\omega - \Delta), & \omega > 0 \\ \partial_\omega N_b(-\omega)\,\Theta(-\omega - \Delta) - N_b(|\omega|)\delta(\omega + \Delta), & \omega < 0 \end{cases} \tag{D-11}$$

According to Eq. (C-17) and following the reasoning used to explain the discontinuity in $R(\varepsilon)$ through Eqs. (C-20) to (C-23), Dirac delta functions in $S^{[1]}(\omega)$ lead to discontinuities at $l\Omega \pm \Delta$ in $f^{[1]}(\epsilon)$ of the system's occupation function at finite bath temperature. These discontinuities in $f^{[1]}(\epsilon)$ will propagate via the same mechanism to $l\Omega \pm n\Delta$ in higher order derivatives $f^{[n]}(\epsilon)$.

As a concrete example, following Eq. (F-5) in the Appendix F, for parabolic bands in 1D and 2D, we have in this case

$$f^{[1]}_{l\Omega^+ \pm\Delta} - f^{[1]}_{l\Omega^- \pm\Delta} \propto \pm\varrho^d(0^+)\Gamma^d_l(0^+, l\Omega^+ \pm \Delta)\Big(f_{0^+}\bar{f}_{l\Omega^+\pm\Delta}N_b(\Delta) - f_{l\Omega^+\pm\Delta}\bar{f}_{0^+}[N_b(\Delta)+1]\Big), \tag{D-12}$$

in which $l\Omega^\pm \equiv l\Omega \pm 0^+$, $\Gamma^d_l(\epsilon_q, \epsilon_p)$ is the scattering amplitude and $\varrho^d(\epsilon) = \epsilon^{(d-2)/2}$ is the density of states, both for the d-dimensional ($d = 1, 2$) parabolic model. The discontinuities in the above first energy derivative $f^{[1]}(\epsilon) = df(\epsilon)/d\epsilon$ can be mapped to discontinuities in $df(k)/dk$ through the chain rule of differentiation.

Appendix E: Dimensionless parabolic models in 1D and 2D

In this section, we non-dimensionalize the parabolic models in one and two dimensions introduced in the main text for the purpose of numerical calculations. We define the relevant dimensionless quantities and relate the particle density to the equilibrium Fermi wave vector and Fermi energy at zero temperature.

For clarity and readability, we omit the bar symbols over dimensionless quantities in the subsequent supplementary materials whenever the context makes it clear that the quantities are dimensionless.

1. Dimensionless parabolic model in 1D

We begin with the 1D parabolic model driven by an oscillating electric field:

$$\epsilon_k = \frac{k^2}{2m} \to \epsilon_k(t) = \frac{[k - A_0\sin(\Omega t + \phi_0)]^2}{2m}. \tag{E-1}$$

To obtain a dimensionless model, we introduce the characteristic energy ϵ_c, and the following dimensionless quantities (note that in the main text we choose $\epsilon_c = \Omega$):

$$\bar{k} = \frac{k}{k_c} = \frac{k}{\sqrt{2m\epsilon_c}}, \quad \bar{\epsilon}_k = \frac{\epsilon_k}{\epsilon_c} = \bar{k}^2, \quad \bar{A}_0 = \frac{A_0}{k_c} = \frac{A_0}{\sqrt{2m\epsilon_c}}, \quad \bar{\Omega} = \frac{\Omega}{\epsilon_c}, \quad \bar{t} = \epsilon_c\, t. \tag{E-2}$$

The dimensionless parameter $\bar{A}_0$ characterizes how far the drive pushes an electron back and forth in the parabolic band relative to $k_c = \sqrt{2m\epsilon_c}$. Using these dimensionless quantities, the 1D parabolic model becomes:

$$\bar{\epsilon}_k = \bar{k}^2 \to \bar{\epsilon}_k(\bar{t}) = [\bar{k} - \bar{A}_0\sin(\bar{\Omega}\bar{t} + \phi_0)]^2. \tag{E-3}$$

Assuming the fermionic system is initially at zero temperature and equilibrium, the particle number is conserved due to the absence of particle exchange with the bosonic bath. We can relate the particle density to the initial equilibrium dimensionless Fermi wave vector $\bar{k}_{F,\text{initial}}$ and Fermi energy $\bar{k}^2_{F,\text{initial}} = \bar{\epsilon}_{F,\text{initial}} = \bar{\mu}_0$, where $\bar{\mu}_0 \equiv \mu_0/\epsilon_c$:

$$\bar{n}_0 = \int_0^{+\infty} d\bar{k}\Theta\left(\bar{\mu}_0 - \bar{k}^2\right) = \sqrt{\bar{\mu}_0} = \bar{k}_{F,\text{initial}}. \tag{E-4}$$

Thus, the particle density n_0 is simply related to the dimensionless Fermi wave vector and energy as $n_0 = \sqrt{\bar{\mu}_0} = \sqrt{\bar{\epsilon}_F} = \bar{k}_F$ in 1D.

The expressions for the Floquet energy harmonics and Floquet wavefunction harmonics for the dimensionless parabolic model in 1D are provided below (with the bar symbol omitted). Floquet energy harmonics:

$$\epsilon_{l,k} = \int_0^T \frac{dt}{T} \epsilon_k(t) e^{+il\Omega t} = [\epsilon_{-l,k}]^*, \tag{E-5}$$

$$\varepsilon_k \equiv \epsilon_{l=0,k} = k^2 + \frac{A_0^2}{2}, \quad \epsilon_{+1,k} = -iA_0 k e^{-i\phi_0}, \quad \epsilon_{+2,k} = -\frac{A_0^2}{4} e^{-2i\phi_0}, \quad \epsilon_{|l|>2,k} = 0. \tag{E-6}$$

Floquet wavefunction harmonics:

$$\varphi_{l,k} = \int_0^T \frac{dt}{T} \left[\exp(+il\Omega t) \exp\left(-\frac{i}{\hbar} \int_0^t dt' \left[\epsilon_k(t') - \varepsilon_k \right] \right) \right]. \tag{E-7}$$

While Floquet wavefunction harmonics lack a closed-form expression, we provide perturbative expressions for $\Gamma_l^{1D}(q,p)$ [see Eq. (C-7)] by calculating the factor $\Phi_{q,p}^{(l)} = \sum_{l_1 \in \mathbf{Z}} |\varphi_{l_1,p} \varphi^*_{l_1+l,q}|^2$ up to $O(A_0^6)$, which is necessary to obtain the scattering matrix $W(q \to p)$:

$$\begin{aligned}
\Gamma_0^{1D}(q,p) &= 2\pi |M_0(q,p)|^2 \left[1 - \frac{2(p-q)^2}{\Omega^2} A_0^2 + \frac{3(p-q)^4}{2\Omega^4} A_0^4 + O(A_0^6) \right], \\
\Gamma_{\pm 1}^{1D}(q,p) &= 2\pi |M_{\pm 1}(q,p)|^2 \left[\frac{(p-q)^2}{\Omega^2} A_0^2 - \frac{(p-q)^4}{\Omega^4} A_0^4 + O(A_0^6) \right], \\
\Gamma_{\pm 2}^{1D}(q,p) &= 2\pi |M_{\pm 2}(q,p)|^2 \left[\frac{(p-q)^4}{4\Omega^4} A_0^4 + O(A_0^6) \right], \\
\Gamma_{\pm 3}^{1D}(q,p) &\propto O(A_0^6).
\end{aligned} \tag{E-8}$$

2. Dimensionless parabolic model in 2D driven by a circularly polarized electric field

The 2D parabolic model driven by a circularly polarized electric field is:

$$\epsilon_{\mathbf{k}} = \frac{\mathbf{k}^2}{2m} \to \epsilon_{\mathbf{k}}^{\pm}(t) = \frac{[k_x - A_0 \sin(\Omega t + \phi_0)]^2}{2m} + \frac{[k_y - A_0 \sin(\Omega t + \phi_0 \pm \pi/2)]^2}{2m}. \tag{E-9}$$

where the superscript $\pm$ denotes the helicity of the circularly polarized electric field: $(+)$ for left-handed (counter-clockwise) and $(-)$ for right-handed (clockwise) polarization. Similar to the previous subsection, we introduce the characteristic energy ϵ_c and the following dimensionless quantities (note that in the main text we choose $\epsilon_c = \Omega$):

$$\bar{k}_{x,y} = \frac{k_{x,y}}{k_c} = \frac{k_{x,y}}{\sqrt{2m\epsilon_c}}, \quad \bar{\epsilon}_{\mathbf{k}} = \frac{\epsilon_{\mathbf{k}}}{\epsilon_c} = \bar{k}_x^2 + \bar{k}_y^2, \quad \bar{A}_0 = \frac{A_0}{k_c} = \frac{A_0}{\sqrt{2m\epsilon_c}}, \quad \bar{\Omega} = \frac{\Omega}{\epsilon_c}, \quad \bar{t} = \epsilon_c\, t. \tag{E-10}$$

The dimensionless 2D parabolic model is then:

$$\bar{\epsilon}_{\mathbf{k}} = \bar{\mathbf{k}}^2 = \bar{k}_x^2 + \bar{k}_y^2 \to \bar{\epsilon}_{\mathbf{k}}^{\pm}(t) = [\bar{k}_x - \bar{A}_0 \sin(\bar{\Omega}\bar{t} + \phi_0)]^2 + [\bar{k}_y - \bar{A}_0 \sin(\bar{\Omega}\bar{t} + \phi_0 \pm \pi/2)]^2. \tag{E-11}$$

In 2D, the relation between the particle density and the initial dimensionless Fermi wave vector and energy is:

$$\bar{n}_0 = \frac{\pi k_{F,\text{initial}}^2}{\pi k_c^2} = \bar{k}_{F,\text{initial}}^2 = \bar{\epsilon}_{F,\text{initial}} = \bar{\mu}_0. \tag{E-12}$$

The expressions for the Floquet energy harmonics and Floquet wavefunction harmonics for the dimensionless parabolic model in 2D are provided below (with the bar symbol omitted).

Floquet energy harmonics:

$$\epsilon_{+l,\mathbf{k}}^{\pm} = \int_0^T \frac{dt}{T} \epsilon_{\mathbf{k}}(t) \exp(+il\Omega t) = [\epsilon_{-l,\mathbf{k}}^{\pm}]^*, \tag{E-13}$$

$$\varepsilon_{\mathbf{k}} \equiv \epsilon^{\pm}_{l=0,\mathbf{k}} = \mathbf{k}^2 + A_0^2, \quad \epsilon^{\pm}_{+1,\mathbf{k}} = A_0 e^{-i\phi_0}(-ik_x \mp k_y), \quad \epsilon^{\pm}_{|l|>1,\mathbf{k}} = 0. \tag{E-14}$$

Floquet wavefunction harmonics can be calculated using the Jacobi-Anger expansion and have closed forms:

$$\begin{aligned}
\varphi^{\pm}_{l,\mathbf{k}} &= \int_0^T \frac{dt}{T}\left[\exp(+il\Omega t)\exp\left(-\frac{i}{\hbar}\int_0^t dt'\left[\epsilon_{\mathbf{k}}(t') - \epsilon^{(0)}_{\mathbf{k}}\right]\right)\right] \\
&= \exp\left(-\sum_{l_1\neq 0}\frac{\epsilon^{(l_1)}_{\mathbf{k}}}{l_1\Omega}\right) \times \int_0^T \frac{dt}{T}\exp\left(\sum_{l_1\neq 0}\frac{\epsilon^{(l_1)}_{\mathbf{k}} e^{-il_1\Omega t}}{l_1\Omega} + il\Omega t\right) \\
&= \exp\left(-\sum_{l_1\neq 0}\frac{\epsilon^{(l_1)}_{\mathbf{k}}}{l_1\Omega}\right) \times J_{\mp l}\left(\frac{2A_0|\mathbf{k}|}{\Omega}\right)\exp(\pm il\theta_{\mathbf{k}} - il\phi_0), \quad \tan\theta_{\mathbf{k}} = k_y/k_x.
\end{aligned} \tag{E-15}$$

Similar to the 1D case, we provide perturbative expressions for $\Gamma^{2\mathrm{D}}_l(\mathbf{q},\mathbf{p})$ [see Eq. (C-7)] by calculating the factor $\Phi^{(l)}_{\mathbf{q},\mathbf{p}} = \sum_{l_1\in\mathbf{Z}} |\varphi_{l_1,\mathbf{p}}\varphi^*_{l_1+l,\mathbf{q}}|^2$ up to $O(A_0^6)$, which is necessary to obtain the scattering matrix $W(\mathbf{q}\to\mathbf{p})$:

$$\begin{aligned}
&\Gamma^{2\mathrm{D}}_0(\mathbf{q},\mathbf{p}) = 2\pi|M_0(\mathbf{q},\mathbf{p})|^2\left[1 - \frac{2(\mathbf{p}^2+\mathbf{q}^2)}{\Omega^2}A_0^2 + \frac{3(\mathbf{p}^4+4\mathbf{p}^2\mathbf{q}^2+\mathbf{p}^4)}{2\Omega^4}A_0^4 + O(A_0^6)\right], \\
&\Gamma^{2\mathrm{D}}_{\pm 1}(\mathbf{q},\mathbf{p}) = 2\pi|M_{\pm 1}(\mathbf{q},\mathbf{p})|^2\left[\frac{(\mathbf{p}^2+\mathbf{q}^2)}{\Omega^2}A_0^2 - \frac{(\mathbf{p}^4+4\mathbf{p}^2\mathbf{q}^2+\mathbf{p}^4)}{\Omega^4}A_0^4 + O(A_0^6)\right], \\
&\Gamma^{2\mathrm{D}}_{\pm 2}(\mathbf{q},\mathbf{p}) = 2\pi|M_{\pm 2}(\mathbf{q},\mathbf{p})|^2\left[\frac{(\mathbf{p}^4+4\mathbf{p}^2\mathbf{q}^2+\mathbf{p}^4)}{4\Omega^4}A_0^4 + O(A_0^6)\right], \\
&\Gamma^{2\mathrm{D}}_{\pm 3}(\mathbf{q},\mathbf{p}) \propto O(A_0^6).
\end{aligned} \tag{E-16}$$

Here we note that $\Gamma^{2\mathrm{D}}_l(\mathbf{q},\mathbf{p})$ does not depend on the helicity of the circularly polarized electric field.

Appendix F: Floquet-Boltzmann equation in energy space for parabolic models with uniform bosonic baths

In this section, we derive Floquet-Boltzmann equations in quasi-energy space for the parabolic models in 1D and 2D. We consider these models coupled to bosonic baths, and for simplicity, we assume the bath has a momentum-independent coupling strength, which we refer to as the uniform coupling approximation, characterized by a constant form factor [see Eq. (6) in the main text]:

$$|\langle q|\chi_\lambda|p\rangle|^2 \to |\chi_0|^2, \tag{F-1}$$

where χ_0 is a constant. This approximation leads to a simplified expression for $\Gamma_l(q,p)$ [see Eq. (C-7)]:

$$\Gamma_l(q,p) \to \Gamma_0 \sum_{l_1\in\mathbf{Z}} |\varphi_{l_1,p}\varphi^*_{l_1+l,q}|^2 = \Gamma_0\Phi^{(l)}_{q,p}, \quad \Gamma_0 \equiv 2\pi|\chi_0|^2, \tag{F-2}$$

as well as a simplified scattering matrix [see Eq. (C-8)]:

$$W(q\to p) = W_e(q\to p) + W_a(q\to p) = \Gamma_0\sum_{l\in\mathbf{Z}}\Phi^{(l)}_{q,p}S(\varepsilon_q - \varepsilon_p + l\Omega), \tag{F-3}$$

which now only depends on the Floquet wavefunction harmonics $\varphi_{l,p}$ and Floquet quasi-energies ε_p.

Before presenting the detailed derivations below, we summarize the Floquet-Boltzmann equation in quasi-energy space for the parabolic models in 1D and 2D that we consider in the main text:

$$0 = \int_0^{+\infty} f(\epsilon_q)W^d_{\epsilon_q\to\epsilon_p}\bar{f}(\epsilon_p)\varrho^d(\epsilon_q)d\epsilon_q - \int_0^{+\infty} f(\epsilon_p)W^d_{\epsilon_p\to\epsilon_q}\bar{f}(\epsilon_q)\varrho^d(\epsilon_q)d\epsilon_q, \quad \bar{f}(\epsilon_p) \equiv 1 - f(\epsilon_p), \tag{F-4}$$

where $f(\epsilon_p)$ is the occupation function labeled by the energy $\epsilon_p = p^2$, $W^d_{\epsilon_p\to\epsilon_q}$ and $\varrho^d(\epsilon_q) = \epsilon_q^{(d-2)/2}$ are the scattering matrix and the density of states for the d-dimensional ($d = 1, 2$) parabolic model in energy space, respectively.

Moreover, following the analysis in the subsection C 2, their n-th derivative with respect to ϵ_p of the occupation reads

$$f^{[n]}_{\epsilon_p} = \sum_{l\in\mathbb{Z}}\sum_{k=0}^{n-1}\sum_{j=0}^{n-k} B^{(njk)}_{\epsilon_p}\int_0^{+\infty} d\epsilon_q \varrho^d(\epsilon_q)\big[\Gamma^d_l(\epsilon_q,\epsilon_p)\big]^{[n-k-j]} \times\Big(f_{\epsilon_q}S^{[j]}(\epsilon_q-\epsilon_p+l\Omega)\bar{f}^{[k]}_{\epsilon_p} - \bar{f}_{\epsilon_q}S^{[j]}(\epsilon_p-\epsilon_q+l\Omega)f^{[k]}_{\epsilon_p}\Big), \tag{F-5}$$

with the coefficient $B^{(njk)}_{\epsilon_p} = \binom{n}{k}\binom{n-k}{j}/R_{\epsilon_p}$, and R_{ϵ_p} represents the maximally allowed scattering rate at momentum p:

$$R_{\epsilon_p} = \int_0^{+\infty} d\epsilon_q \varrho^d(\epsilon_q)\big(f_{\epsilon_q}W^d_{\epsilon_q,\epsilon_p} + W^d_{\epsilon_p,\epsilon_q}\bar{f}_{\epsilon_q}\big) \tag{F-6}$$

In the following sub-sections, we provide detailed derivations of the Floquet-Boltzmann equation for the 1D and 2D parabolic models coupled to simple bosonic baths under the uniform coupling approximation. The derivations lead to the specific scattering matrix $W^d_{\epsilon_p\to\epsilon_q} = \sum_l \Gamma^d_l(\epsilon_q,\epsilon_p)S(\epsilon_q-\epsilon_p+l\Omega)$ and density of states $\varrho^d(\epsilon_q)$ used in Eqs. (F-4), (F-5), and (F-6).

1. Floquet-Boltzmann equation for the 1D parabolic model

For the 1D parabolic model coupled to the bosonic bath described above, the Floquet-Boltzmann equation [see Eq. (5) in the main text] takes the form:

$$0 = \int_{-\infty}^{+\infty} f_q W_{q\to p}\bar{f}_p \frac{dq}{2\pi} - \int_{-\infty}^{+\infty} f_p W_{p\to q}\bar{f}_q \frac{dq}{2\pi}, \tag{F-7}$$

with the scattering matrix given by Eq. (F-3). Given the momentum-independent difference between the original dispersion $\epsilon_k = k^2$ and the quasi-energy $\varepsilon_k = k^2 + A_0^2/2$ for the 1D parabolic model, we can recast the momentum-space equation in terms of the energy variable ϵ_k:

$$0 = \int_0^{+\infty} f(\epsilon_q)W^{1\mathrm{D}}_{\epsilon_q\to\epsilon_p}\bar{f}(\epsilon_p)\frac{d\epsilon_q}{\sqrt{\epsilon_q}} - \int_0^{+\infty} f(\epsilon_p)W^{1\mathrm{D}}_{\epsilon_p\to\epsilon_q}\bar{f}(\epsilon_q)\frac{d\epsilon_q}{\sqrt{\epsilon_q}}, \tag{F-8}$$

in which we define the summed scattering matrix as

$$W^{1\mathrm{D}}_{\epsilon_q\to\epsilon_p} \equiv W_{-\sqrt{\epsilon_q}\to+\sqrt{\epsilon_p}} + W_{+\sqrt{\epsilon_q}\to+\sqrt{\epsilon_p}}, \quad W^{1\mathrm{D}}_{\epsilon_p\to\epsilon_q} \equiv W_{+\sqrt{\epsilon_p}\to-\sqrt{\epsilon_q}} + W_{+\sqrt{\epsilon_p}\to+\sqrt{\epsilon_q}} \tag{F-9}$$

and use the symmetry $f(\epsilon_p) = f(+p) = f(-p)$ to obtain the above equation. Eq. (F-8) is a specialized case of the general Eq. (C-11).

Using Eqs. (E-8) and (F-3), we write the summed scattering matrix up to $O(A_0^6)$:

$$\begin{aligned} W^{1\mathrm{D}}_{\epsilon_q\to\epsilon_p} &= \sum_l \Gamma^{1\mathrm{D}}_l(\epsilon_q,\epsilon_p)S(\epsilon_q-\epsilon_p+l\Omega) \\ &= \Gamma_0\Big[2 - \frac{4(\epsilon_p+\epsilon_q)}{\Omega^2}A_0^2 + \frac{3(\epsilon_p^2+6\epsilon_p\epsilon_q+\epsilon_q^2)}{\Omega^4}A_0^4\Big]S(\epsilon_q-\epsilon_p) \\ &\quad + \sum_{\eta=\pm1}\Gamma_0\Big[\frac{2(\epsilon_p+\epsilon_q)}{\Omega^2}A_0^2 - \frac{2(\epsilon_p^2+6\epsilon_p\epsilon_q+\epsilon_q^2)}{\Omega^4}A_0^4\Big]S(\epsilon_q-\epsilon_p+\eta\Omega) \\ &\quad + \sum_{\eta=\pm1}\Gamma_0\Big[\frac{(\epsilon_p^2+6\epsilon_p\epsilon_q+\epsilon_q^2)}{2\Omega^4}A_0^4\Big]S(\epsilon_q-\epsilon_p+2\eta\Omega) + O(A_0^6), \end{aligned} \tag{F-10}$$

while $W^{1\mathrm{D}}_{\epsilon_p\to\epsilon_q}$ is obtained by swapping $\epsilon_q \leftrightarrow \epsilon_p$ in the above expression.

Here we also show the formal expressions of derivatives of $f(\epsilon_p)$ following the analysis in the subsection C 2. We take the n-th derivative with respect to ϵ_p on both sides of Eq. (F-8), apply the general Leibniz rule, and obtain:

$$f^{[n]}_{\epsilon_p} = \frac{1}{R_{\epsilon_p}}\sum_{k=0}^{n-1}\binom{n}{k}\int_0^{+\infty}\frac{d\epsilon_q}{\sqrt{\epsilon_q}}\Big(f_{\epsilon_q}\big[W^{1\mathrm{D}}_{\epsilon_q\to\epsilon_p}\big]^{[n-k]}\bar{f}^{[k]}_{\epsilon_p} - f^{[k]}_{\epsilon_p}\big[W^{1\mathrm{D}}_{\epsilon_p\to\epsilon_q}\big]^{[n-k]}\bar{f}_{\epsilon_q}\Big), \tag{F-11}$$

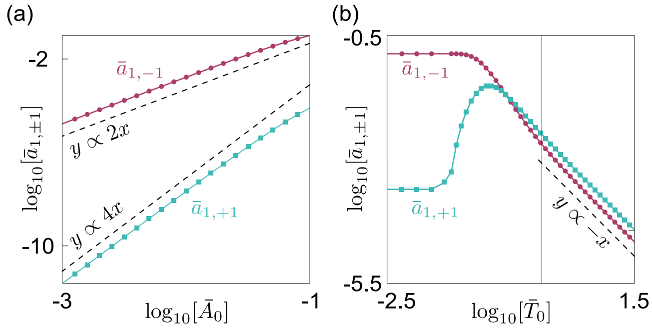

FIG. F-4. Dependence of $\bar{a}_{n=1,s=+1} = \Omega a_{n=1,s=+1}$ (green lines) and $\bar{a}_{n=1,s=-1} = \Omega a_{n=1,s=-1}$ (purple lines) as functions of (a) driving amplitude $\bar{A}_0 = A_0/\sqrt{2m\Omega}$ and (b) bath temperature $\bar{T}_0 = k_B T_0/\Omega$ for the 2D parabolic model [see Eq. (8)] coupled to the gapped bosonic bath [see Eq. (11)]. Parameters used both panels: $\Delta/\Omega = 3/10$, particle density $n_0/(2m\Omega) = 3/13$. For panel (a), bath temperature $T_0 = 0$. For panel (b), driving amplitude $A_0/\sqrt{6m\Omega} = 1/5$.

where the superscript $[k]$ on a function denotes its k-th derivative with respect to the energy ϵ_p, $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the binomial coefficient, and R_{ϵ_p} represents the maximally allowed scattering rate at momentum p:

$$R_{\epsilon_p} = \int_0^{+\infty} \frac{d\epsilon_q}{\sqrt{\epsilon_q}} \left(f_{\epsilon_q} W^{1\mathrm{D}}_{\epsilon_q,\epsilon_p} + W^{1\mathrm{D}}_{\epsilon_p,\epsilon_q} \bar{f}_{\epsilon_q} \right) \tag{F-12}$$

We further apply the general Leibniz rule on $W^{[n-k]}_{\epsilon_q\to\epsilon_p}$ and $W^{[n-k]}_{\epsilon_p\to\epsilon_q}$ [see Eq. (F-10)]:

$$\begin{aligned}
\left[W^{1\mathrm{D}}_{\epsilon_q\to\epsilon_p}\right]^{[n-k]} &= \sum_{l\in\mathbb{Z}} \sum_{j=0}^{n-k} \binom{n-k}{j} \left[\Gamma_l^{1\mathrm{D}}(\varepsilon_q,\epsilon_p)\right]^{[n-k-j]} S^{[j]}(\epsilon_q - \epsilon_p + l\Omega), \\
\left[W^{1\mathrm{D}}_{\epsilon_p\to\epsilon_q}\right]^{[n-k]} &= \sum_{l\in\mathbb{Z}} \sum_{j=0}^{n-k} \binom{n-k}{j} \left[\Gamma_l^{1\mathrm{D}}(\epsilon_q,\epsilon_p)\right]^{[n-k-j]} S^{[j]}(\epsilon_p - \epsilon_p + l\Omega).
\end{aligned} \tag{F-13}$$

Combining these results, we obtain:

$$\begin{aligned}
f^{[n]}_{\epsilon_p} &= \frac{1}{R_{\epsilon_p}} \sum_{l\in\mathbb{Z}} \sum_{k=0}^{n-1} \sum_{j=0}^{n-k} \binom{n}{k} \binom{n-k}{j} \int_0^{+\infty} \frac{d\epsilon_q}{\sqrt{\epsilon_q}} \left[\Gamma_l^{1\mathrm{D}}(\epsilon_q,\epsilon_p)\right]^{[n-k-j]} \left(f_{\epsilon_q} S^{[j]}_{\epsilon_q-\epsilon_p+l\Omega} \bar{f}^{[k]}_{\epsilon_p} - \bar{f}_{\epsilon_q} S^{[j]}_{\epsilon_p-\epsilon_q+l\Omega} f^{[k]}_{\epsilon_p} \right) \\
&= \sum_{l\in\mathbb{Z}} \sum_{k=0}^{n-1} \sum_{j=0}^{n-k} B^{(njk)}_{\epsilon_p} \int_0^{+\infty} \frac{d\epsilon_q}{\sqrt{\epsilon_q}} \left[\Gamma_l^{1\mathrm{D}}(\epsilon_q,\epsilon_p)\right]^{[n-k-j]} \left(f_{\epsilon_q} S^{[j]}(\epsilon_q - \epsilon_p + l\Omega) \bar{f}^{[k]}_{\epsilon_p} - \bar{f}_{\epsilon_q} S^{[j]}(\epsilon_p - \epsilon_q + l\Omega) f^{[k]}_{\epsilon_p} \right),
\end{aligned} \tag{F-14}$$

with the coefficient $B^{(njk)}_{\epsilon_p} = \binom{n}{k}\binom{n-k}{j}/R_{\epsilon_p}$.

2. Floquet-Boltzmann equation for the 2D parabolic model driven by a circularly polarized electric field

For the 2D parabolic model driven by a circularly polarized electric field, the Floquet-Boltzmann equation takes the form:

$$0 = \int_{-\infty}^{+\infty} f_{\mathbf{q}} W_{\mathbf{q}\to\mathbf{p}} \bar{f}_{\mathbf{p}} \frac{d\mathbf{q}}{(2\pi)^2} - \int_{-\infty}^{+\infty} f_{\mathbf{p}} W_{\mathbf{p}\to\mathbf{q}} \bar{f}_{\mathbf{q}} \frac{d\mathbf{q}}{(2\pi)^2}, \tag{F-15}$$

with the scattering matrix given by Eq. (F-3). Given the momentum-independent difference between the original dispersion $\epsilon_k \equiv \epsilon_{\mathbf{k}} = \mathbf{k}^2$ and the quasi-energy $\varepsilon_k \equiv \varepsilon_{\mathbf{k}} = \mathbf{k}^2 + A_0^2$ for the 2D parabolic model, we can recast the momentum-space equation in terms of the energy variable ϵ_k:

$$0 = \int_0^{+\infty} f(\epsilon_q) W^{2\mathrm{D}}_{\epsilon_q\to\epsilon_p} \bar{f}(\epsilon_p) d\epsilon_q - \int_0^{+\infty} f(\epsilon_p) W^{2\mathrm{D}}_{\epsilon_p\to\epsilon_q} \bar{f}(\epsilon_q) d\epsilon_q, \tag{F-16}$$

in which we define the scattering matrix as

$$W^{2D}_{\epsilon_q\to\epsilon_p} \equiv W_{\sqrt{\epsilon_q}\to\sqrt{\epsilon_p}}, \quad W^{2D}_{\epsilon_p\to\epsilon_q} \equiv W_{\sqrt{\epsilon_p}\to\sqrt{\epsilon_q}} \tag{F-17}$$

by using the fact that $W_{\mathbf{q}\to\mathbf{p}}$ is a function of $\{\epsilon_p = \mathbf{p}^2, \epsilon_q = \mathbf{q}^2\}$ and does not depend on the orientations of $\mathbf{p}$ and $\mathbf{q}$ [see Eq. (E-16)] and $f(\mathbf{p}) = f(|\mathbf{p}|)$, for the 2D parabolic model driven by a circularly polarized electric field.

Using Eqs. (E-16) and (F-3), we write the scattering matrix up to $O(A_0^6)$:

$$\begin{aligned} W^{2D}_{\epsilon_q\to\epsilon_p} &= \sum_l \Gamma^{2D}_l(\epsilon_q,\epsilon_p) S(\epsilon_q-\epsilon_p+l\Omega) \\ &= \Gamma_0\Big[1-\frac{2(\epsilon_p+\epsilon_q)}{\Omega^2}A_0^2+\frac{3(\epsilon_p^2+4\epsilon_p\epsilon_q+\epsilon_q^2)}{2\Omega^4}A_0^4\Big]S(\epsilon_q-\epsilon_p) \\ &\quad+\sum_{\eta=\pm1}\Gamma_0\Big[\frac{(\epsilon_p+\epsilon_q)}{\Omega^2}A_0^2-\frac{(\epsilon_p^2+4\epsilon_p\epsilon_q+\epsilon_q^2)}{\Omega^4}A_0^4\Big]S(\epsilon_q-\epsilon_p+\eta\Omega) \\ &\quad+\sum_{\eta=\pm1}\Gamma_0\Big[\frac{(\epsilon_p^2+4\epsilon_p\epsilon_q+\epsilon_q^2)}{4\Omega^4}A_0^4\Big]S(\epsilon_q-\epsilon_p+2\eta\Omega)+O(A_0^6), \end{aligned} \tag{F-18}$$

while $W^{2D}_{\epsilon_p\to\epsilon_q}$ is obtained by swapping $\epsilon_q \leftrightarrow \epsilon_p$ in the above expression.

Here we also show the formal expressions of derivatives of $f(\epsilon_p)$ following the analysis in the subsection C 2. We take the n-th derivative with respect to ϵ_p on both sides of Eq. (F-16), apply the general Leibniz rule, and obtain:

$$f^{[n]}_{\epsilon_p} = \frac{1}{R_{\epsilon_p}}\sum_{k=0}^{n-1}\binom{n}{k}\int_0^{+\infty} d\epsilon_q\Big(f_{\epsilon_q}\big[W^{2D}_{\epsilon_q\to\epsilon_p}\big]^{[n-k]}\bar{f}^{[k]}_{\epsilon_p} - f^{[k]}_{\epsilon_p}\big[W^{2D}_{\epsilon_p\to\epsilon_q}\big]^{[n-k]}\bar{f}_{\epsilon_q}\Big), \tag{F-19}$$

where the superscript $[k]$ on a function denotes its k-th derivative with respect to the energy ϵ_p, $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the binomial coefficient, and R_{ϵ_p} represents the maximally allowed scattering rate at momentum p:

$$R_{\epsilon_p} = \int_0^{+\infty} d\epsilon_q\big(f_{\epsilon_q}W^{2D}_{\epsilon_q,\epsilon_p}+W^{2D}_{\epsilon_p,\epsilon_q}\bar{f}_{\epsilon_q}\big) \tag{F-20}$$

We further apply the general Leibniz rule on $W^{[n-k]}_{\epsilon_q\to\epsilon_p}$ and $W^{[n-k]}_{\epsilon_p\to\epsilon_q}$ [see Eq. (F-18)]:

$$\begin{aligned} \big[W^{2D}_{\epsilon_q\to\epsilon_p}\big]^{[n-k]} &= \sum_{l\in\mathbb{Z}}\sum_{j=0}^{n-k}\binom{n-k}{j}\big[\Gamma^{2D}_l(\varepsilon_q,\epsilon_p)\big]^{[n-k-j]}S^{[j]}(\epsilon_q-\epsilon_p+l\Omega), \\ \big[W^{2D}_{\epsilon_p\to\epsilon_q}\big]^{[n-k]} &= \sum_{l\in\mathbb{Z}}\sum_{j=0}^{n-k}\binom{n-k}{j}\big[\Gamma^{2D}_l(\epsilon_q,\epsilon_p)\big]^{[n-k-j]}S^{[j]}(\epsilon_p-\epsilon_p+l\Omega). \end{aligned} \tag{F-21}$$

Combining these results, we obtain:

$$\begin{aligned} f^{[n]}_{\epsilon_p} &= \frac{1}{R_{\epsilon_p}}\sum_{l\in\mathbb{Z}}\sum_{k=0}^{n-1}\sum_{j=0}^{n-k}\binom{n}{k}\binom{n-k}{j}\int_0^{+\infty} d\epsilon_q\big[\Gamma^{2D}_l(\epsilon_q,\epsilon_p)\big]^{[n-k-j]}\Big(f_{\epsilon_q}S^{[j]}_{\epsilon_q-\epsilon_p+l\Omega}\bar{f}^{[k]}_{\epsilon_p}-\bar{f}_{\epsilon_q}S^{[j]}_{\epsilon_p-\epsilon_q+l\Omega}f^{[k]}_{\epsilon_p}\Big) \\ &= \sum_{l\in\mathbb{Z}}\sum_{k=0}^{n-1}\sum_{j=0}^{n-k}B^{(njk)}_{\epsilon_p}\int_0^{+\infty} d\epsilon_q\big[\Gamma^{2D}_l(\epsilon_q,\epsilon_p)\big]^{[n-k-j]}\Big(f_{\epsilon_q}S^{[j]}(\epsilon_q-\epsilon_p+l\Omega)\bar{f}^{[k]}_{\epsilon_p}-\bar{f}_{\epsilon_q}S^{[j]}(\epsilon_p-\epsilon_q+l\Omega)f^{[k]}_{\epsilon_p}\Big), \end{aligned} \tag{F-22}$$

with the coefficient $B^{(njk)}_{\epsilon_p} = \binom{n}{k}\binom{n-k}{j}/R_{\epsilon_p}$.

Appendix G: Equal time pair correlation function for the Floquet non-Fermi liquid

In this section, we first derive the pair correlation function for a periodically driven Bloch band and then demonstrate that it exhibits long-range or power-law behavior for the Floquet non-Fermi liquid using specific examples.

The pair correlation function quantifies the probability density of finding a particle at position $\mathbf{r}_1$ given that another particle is located at position $\mathbf{r}_2$, making it a crucial characterization of the correlations in a fluid. The correlation function can be derived as follows [25]

$$g(\mathbf{r}_1,\mathbf{r}_2,t)=\frac{\langle n(\mathbf{r}_1,t)n(\mathbf{r}_2,t)\rangle}{\langle n(\mathbf{r}_1,t)\rangle\,\langle n(\mathbf{r}_2,t)\rangle}-\frac{\delta(\mathbf{r}_2-\mathbf{r}_1)}{\langle n(\mathbf{r}_1,t)\rangle}=1-\frac{\langle c^\dagger_{\mathbf{r}_1}(t)c_{\mathbf{r}_2}(t)\rangle\,\langle c^\dagger_{\mathbf{r}_2}(t)c_{\mathbf{r}_1}(t)\rangle}{n_0^2}, \tag{G-1}$$

where Wick's theorem is applied, and the system is assumed to be translationally invariant, i.e., $\langle n(\mathbf{r}_1,t_1)\rangle = \langle n(\mathbf{r}_2,t_2)\rangle = n_0$, which represents the averaged uniform density. For the single-band model under consideration, the creation/annihilation operators in Eq. (G-1) can be expanded in terms of plane waves as follows:

$$c^\dagger_{\mathbf{r}}(t)=\frac{1}{\sqrt{V}}\sum_{\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{r}}[\psi_{\mathbf{k}}(t)]^*c^\dagger_{\mathbf{k}}, \tag{G-2}$$

where $\psi_{\mathbf{k}}(t)$ denotes the Floquet wave function [see Eq. (B-2)]. In the Floquet non-Fermi liquid steady state, we have $\langle c^\dagger_{\mathbf{k}'}c_{\mathbf{k}}\rangle=\delta_{\mathbf{k}'\mathbf{k}}f_{\mathbf{k}}$. By denoting $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$, we obtain the following expression for Eq. (G-1):

$$g(\mathbf{r})=1-\frac{1}{n_0^2}\left|\frac{1}{V}\sum_{\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{r}}f_{\mathbf{k}}\right|^2=1-\left|\frac{\sum_{\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{r}}f_{\mathbf{k}}/V}{\sum_{\mathbf{k}}f_{\mathbf{k}}/V}\right|^2=1-\left|\frac{\tilde f(\mathbf{r})}{\tilde f(\mathbf{0})}\right|^2, \tag{G-3}$$

where $\tilde f(\mathbf{r})$ is defined as the d-dimensional Fourier transform of $f_{\mathbf{k}}$:

$$\tilde f(\mathbf{r})=\int\frac{d\mathbf{k}}{(2\pi)^d}e^{-i\mathbf{k}\cdot\mathbf{r}}f_{\mathbf{k}}. \tag{G-4}$$

In a conventional equilibrium Fermi liquid at zero temperature, i.e., $T_0=0$, $f_{\mathbf{k}}\to\Theta(k_F-|\mathbf{k}|)$. The discontinuity in $f_{\mathbf{k}}$ at the Fermi surface causes long-range oscillatory behavior in $g(\mathbf{r})$, which is determined by the Fermi wave vector k_F and the system dimensionality [25]. In contrast, the Floquet non-Fermi liquid exhibits power-law long-range oscillations originating from non-analyticities in its occupation function $f_{\mathbf{k}}$, even at finite temperatures. This is demonstrated next for 1D and 2D parabolic models.

1. Parabolic model in 1D

We now evaluate the integral $\int dk e^{-ikr}f_k/(2\pi)$ from Eq. (G-4) for the dimensionless parabolic model in 1D [see Eq. (E-3)]. Our focus is on the square-root-Theta type non-analyticities in the momentum distribution at specific momenta q_j ($j=1,2,\ldots$), which arise from the coupling with gapped bosonic baths [see Eq. (11) in the main text]. The momentum distribution can be decomposed as follows:

$$f^{\{1\}}(k)=f^{\{1\}}_{\text{regular}}(k)+\sum_j\frac{a_j\Theta(k-q_j)}{(k-q_j)^{1/2}}, \tag{G-5}$$

where the superscript $\{k\}$ denotes the k-th derivative with respect to momentum p, $f^{\{1\}}_{\text{regular}}(k)$ encompasses the analytic component and any non-analyticities weaker than $\Theta(k-q_j)(k-q_j)^{-1/2}$ with $\Theta(x)$ being the Heaviside step function.

Exploiting the symmetry property $f(-k)=f(k)$, which implies $f^{\{1\}}(-k)=-f^{\{1\}}(k)$, and applying integration by parts, we obtain:

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{+\infty}dk\,e^{-ikr}f_k&=-\frac{1}{2\pi}\int_0^{+\infty}dk\,\frac{2\sin(kr)}{r}f^{\{1\}}(k)=-\frac{1}{2\pi}\int_0^{+\infty}dk\,\frac{2\sin(kr)}{r}\Big[f^{\{1\}}_{\text{regular}}(k)+\sum_j\frac{a_j\Theta(k-q_j)}{(k-q_j)^{1/2}}\Big]\\&=\text{regular part}-\frac{1}{r^{3/2}}\sum_j\sqrt{\frac{2}{\pi}}a_j\Big[\cos(q_jr)S_F\Big(\sqrt{(2/\pi)\,(k-q_j)\,r}\Big)+\sin(q_jr)C_F\Big(\sqrt{(2/\pi)\,(k-q_j)\,r}\Big)\Big]_{q_j}^{+\infty}\\&=\text{regular part}-\frac{1}{r^{3/2}}\sum_j\frac{a_j}{\sqrt{\pi}}\sin\Big(q_jr+\frac{\pi}{4}\Big),\end{aligned} \tag{G-6}$$

where $S_F(z)$ and $C_F(z)$ are Fresnel integrals, and their asymptotic properties $S_F(0)=C_F(0)=0$ and $S_F(+\infty)=C_F(+\infty)=1/2$ were used. The non-analyticities thus lead to power-law decay $\sim r^{-3/2}$ with oscillations at Floquet Fermi surfaces q_j, contrasting the behavior of conventional Fermi liquids.

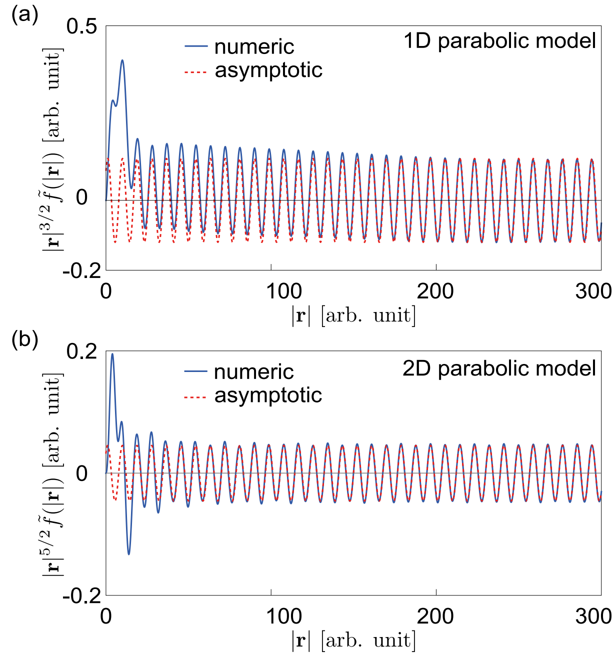

FIG. G-5. Re-scaled real space Fourier transform $\tilde{f}(\mathbf{r})$ [Eq. (G-4)] obtained from direct numerical solutions of $f(\mathbf{k})$ (blue solid lines) and asymptotic analysis (red dashed lines) for (a) a one-dimensional parabolic model [Eq. (G-6)] and (b) a two-dimensional parabolic model [Eq. (G-8)] coupled with a gapped bosonic bath with $T_0 = 0$ and $\Delta = 0$ such that there is only one primary Fermi surface at Ω, for visual clarity. The asymptotic behavior of $\tilde{f}(\mathbf{r})$ for both models can be described by a unified expression, as discussed in Eq. (13) of the main text. The following parameters are used: $A_0/\sqrt{m\Omega} = 4/5$, particle density $n_0/\sqrt{m\Omega} = 1/2$ (1D) or $n_0/(m\Omega) = 1$ (2D).

2. Parabolic model in 2D

Here we evaluate the integral $\int d\mathbf{k}e^{-i\mathbf{k}\cdot\mathbf{r}} f_{\mathbf{k}}/(2\pi)^2$ from Eq. (G-4) for the dimensionless parabolic model in 2D [see Eq. (E-11)]. Our focus is on the Theta-type non-analyticities in the momentum distribution at specific momenta p_i ($i = 1, 2, \ldots$), which arise from the coupling with gapped bosonic baths [see Eq. (11) in the main text]. The momentum distribution can be decomposed as follows:

$$f^{\{1\}}(k) = f^{\{1\}}_{\text{regular}}(k) + \sum_i a_i \Theta(k - p_i), \tag{G-7}$$

where the superscript $\{k\}$ denotes the k-th derivative with respect to momentum p, $f^{\{1\}}_{\text{regular}}(k)$ encompasses the analytic component and any non-analyticities weaker than $\Theta(k - p_i)$. Exploiting the symmetry property $f(\mathbf{k}) = f(k)$ with $k = |\mathbf{k}|$, and applying integration by parts, we obtain:

$$\begin{aligned}
&\frac{1}{(2\pi)^2}\int d\mathbf{k}\, e^{-i\mathbf{k}\cdot\mathbf{r}} f_{\mathbf{k}} = \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty} kdk \int_0^{2\pi} d\theta e^{-ikr\cos\theta} f(k) = -\frac{1}{2\pi}\int_0^{+\infty} dk\, \frac{kJ_1(kr)}{r} f^{\{1\}}(k) \\
&= -\frac{1}{2\pi}\int_0^{+\infty} dk\, \frac{kJ_1(kr)}{r}\Big[f^{\{1\}}_{\text{regular}}(k) + \sum_i a_i\Theta(k - p_i)\Big] \\
&= -\frac{1}{2\pi}\int_0^{+\infty} dk\, \frac{kJ_1(kr)}{r}\Big[\mathring{f}^{\{1\}}_{\text{regular}}(k) + \sum_i a_i\Theta(p_i - k)\Big] \\
&= \text{regular part} - \frac{1}{2\pi}\sum_i a_i \frac{k\pi\big[J_1(kr)H_0(kr) - J_0(kr)H_1(kr)\big]}{2r^2}\Bigg|_0^{p_i} \\
&\xrightarrow{r\to+\infty} \text{regular part} - \frac{1}{2\pi}\frac{1}{r^{5/2}}\sum_i a_i\sqrt{\frac{2}{\pi}}p_i \sin\Big(p_i r + \frac{\pi}{4}\Big)\,.
\end{aligned} \tag{G-8}$$

We note that in the above derivations, we define $\mathring{f}^{\{1\}}_{\text{regular}}(k)$ such that $\mathring{f}^{\{1\}}_{\text{regular}}(k) + \sum_i a_i \Theta(p_i - k) = f^{\{1\}}_{\text{regular}}(k) + \sum_i a_i \Theta(k - p_i)$. Similar to $\lim_{k\to\infty} f^{\{1\}}(k) = 0$, we have $\lim_{k\to\infty} \mathring{f}^{\{1\}}_{\text{regular}}(k) = 0$. Consequently, the integral with respect to $\mathring{f}^{\{1\}}_{\text{regular}}(k)$ is convergent. The "regular part" refers to the part whose oscillation decays faster than $r^{-5/2}$ at large r, $J_{0,1}(x)$ and $H_{0,1}(x)$ are Bessel and Struve functions. The non-analyticities produce power-law decay $\sim r^{-5/2}$ with oscillations at Floquet Fermi surfaces p_i, again distinct from conventional Fermi liquids.

Appendix H: Density noise correlation function for the Floquet non-Fermi liquid

In this section, we derive the density noise correlation function for a periodically driven Bloch band and provide explicit expressions for the parabolic models in 1D and 2D.

The spectrum of density fluctuations can be obtained from the correlation function

$$C(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2) = \langle n(\mathbf{r}_1, t_1) n(\mathbf{r}_2, t_2)\rangle - \langle n(\mathbf{r}_1, t_1)\rangle \langle n(\mathbf{r}_2, t_2)\rangle . \tag{H-1}$$

Due to translational invariance, $\langle n(\mathbf{r}_1, t_1)\rangle = \langle n(\mathbf{r}_2, t_2)\rangle = n_0$. Thus, the non-trivial part is the correlation function:

$$\langle n(\mathbf{r}_1, t_1) n(\mathbf{r}_2, t_2)\rangle = \langle c^\dagger_{\mathbf{r}_1}(t_1) c_{\mathbf{r}_1}(t_1) c^\dagger_{\mathbf{r}_2}(t_2) c_{\mathbf{r}_2}(t_2)\rangle . \tag{H-2}$$

Using the relation in Eq. (G-2) and performing contractions $\langle c^\dagger_{\mathbf{k}_1} c_{\mathbf{k}_2} c^\dagger_{\mathbf{k}_3} c_{\mathbf{k}_4}\rangle = \langle c^\dagger_{\mathbf{k}_1} c_{\mathbf{k}_2}\rangle \langle c^\dagger_{\mathbf{k}_3} c_{\mathbf{k}_4}\rangle + \langle c^\dagger_{\mathbf{k}_1} c_{\mathbf{k}_4}\rangle \langle c_{\mathbf{k}_2} c^\dagger_{\mathbf{k}_3}\rangle$ according to Wick's theorem, we obtain

$$C(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2) = \frac{1}{V^2} \sum_{\mathbf{k}_1, \mathbf{k}_2} f_{\mathbf{k}_1} \bar{f}_{\mathbf{k}_2} e^{-i(\mathbf{k}_1 - \mathbf{k}_2)\cdot(\mathbf{r}_1 - \mathbf{r}_2)} [\psi_{\mathbf{k}_1}(t_1)]^* \psi_{\mathbf{k}_2}(t_1) [\psi_{\mathbf{k}_2}(t_2)]^* \psi_{\mathbf{k}_1}(t_2). \tag{H-3}$$

By Fourier transforming into momentum space, we have

$$C(\mathbf{q}; t_1, t_2) = \frac{1}{V} \sum_{\mathbf{k}} f_{\mathbf{k}+\mathbf{q}} \bar{f}_{\mathbf{k}} [\psi_{\mathbf{k}+\mathbf{q}}(t_1)]^* \psi_{\mathbf{k}}(t_1) [\psi_{\mathbf{k}}(t_2)]^* \psi_{\mathbf{k}+\mathbf{q}}(t_2). \tag{H-4}$$

Averaging over t_2, Fourier transforming $t = t_1 - t_2$ into frequency space, and performing Floquet expansions [see Eq. (B-2)], we obtain

$$\bar{C}(\mathbf{q}; \omega) \equiv \int_0^T \frac{dt_2}{T} \left[\int_{-\infty}^{+\infty} \frac{dt}{2\pi} C(\mathbf{q}; t_1, t_2) e^{+i\omega t}\right] = \frac{1}{V} \sum_{\mathbf{k}} f_{\mathbf{k}+\mathbf{q}} \bar{f}_{\mathbf{k}} \sum_l \Phi_l(\mathbf{k}, \mathbf{k}+\mathbf{q}) \delta(\varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{k}+\mathbf{q}} - \omega + l\Omega), \tag{H-5}$$

where the l-dependent amplitude factor reads

$$\Phi_l(\mathbf{k}, \mathbf{k}+\mathbf{q}) = \Big| \sum_{l_1} \varphi_{l,\mathbf{k}} \varphi^*_{l+l_1, \mathbf{k}+\mathbf{q}} \Big|^2 . \tag{H-6}$$

In equilibrium, $\Phi_l(\mathbf{k}, \mathbf{k}+\mathbf{q}) = \delta_{l0}$ and the density noise correlation function reduces to the dynamic structure factor:

$$\bar{C}(\mathbf{q}; \omega) \to C_{\text{equi}}(\mathbf{q}; \omega) = \frac{1}{V} \sum_{\mathbf{k}} f_{\mathbf{k}+\mathbf{q}} \bar{f}_{\mathbf{k}} \delta(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}+\mathbf{q}} - \omega). \tag{H-7}$$

At zero temperature, the dynamic structure factor $C_{\text{equi}}(\mathbf{q}; \omega)$ is non-zero only when $\mathbf{q}$ and ω lie within the particle-hole continuum. At finite temperatures, this region smoothly broadens due to thermal excitations. However, as we will demonstrate, this is not the case for our Floquet non-Fermi liquid, where the particle-hole continuum remains sharply defined even at finite temperatures.

1. Parabolic model in 1D

We now evaluate explicitly the density noise correlation function, Eq. (H-5) for the parabolic model in 1D [see Eq. (E-3)]. Given the quasi-energy $\varepsilon_k = k^2 + A_0^2/2$ for the 1D parabolic model, we obtain

$$\bar{C}_{\text{1D}}(q; \omega) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} f_{k+q} \bar{f}_k \sum_l \Phi_l(k, k+q) \delta\big(k^2 - (k+q)^2 - \omega + l\Omega\big). \tag{H-8}$$

The l-dependent amplitude $\Phi_l(k,k+q) = |\sum_{l_1} \varphi_{l,k}\varphi^*_{l+l_1,k+q}|^2 \to \Phi_l(q)$ is k-independent for the 1D parabolic model and can be read from $\Gamma_l^{\rm 1D}(k,k+q)$ [see Eq. (E-8)]:

$$\begin{aligned}
\Phi_{l=0}(q) &= 1 - \frac{2q^2}{\Omega^2}A_0^2 + \frac{3q^4}{2\Omega^4}A_0^4 + O(A_0^6),\\
\Phi_{l=\pm1}(q) &= \frac{q^2}{\Omega^2}A_0^2 - \frac{q^4}{\Omega^4}A_0^4 + O(A_0^6),\\
\Phi_{l=\pm2}(q) &= \frac{q^4}{4\Omega^4}A_0^4 + O(A_0^6),\\
\Phi_{l=\pm3}(q) &\propto O(A_0^6).
\end{aligned} \tag{H-9}$$

Evaluating the integral in Eq. (H-8) by eliminating the Dirac delta function,

$$\delta\big(k^2 - (k+q)^2 - \omega + l\Omega\big) \to \frac{1}{2|q|}\delta\left(k - \frac{-q^2-\omega+l\Omega}{2q}\right) \tag{H-10}$$

we arrive at the expression

$$\bar{C}_{\rm 1D}(q;\omega) = \frac{1}{4\pi|q|}\sum_l \Phi_l(q) f\left(\frac{-q^2-\omega+l\Omega}{2q} + q\right)\bar{f}\left(\frac{-q^2-\omega+l\Omega}{2q}\right). \tag{H-11}$$

2. Parabolic model in 2D

We now explicitly evaluate the density noise correlation function, Eq. (H-5), for the parabolic model in 2D [see Eq. (E-11)]. Given the quasi-energy $\varepsilon_{\mathbf{k}} = \mathbf{k}^2 + A_0^2$, the rotationally invariant occupation function $f(|\mathbf{k}|)$, and the closed form for its Floquet wave function harmonics [see Eq. (E-15)], we have:

$$\begin{aligned}
\bar{C}^{\pm}_{\rm 2D}(q,\theta_q;\omega) = \frac{1}{(2\pi)^2}\int_0^{2\pi} d\theta_k \int_0^{+\infty} kdk\, f(|\mathbf{k}+\mathbf{q}|)\bar{f}(k)\sum_l \left|\sum_{l_1} J_l\left(\frac{2A_0k}{\Omega}\right) J_{l+l_1}\left(\frac{2A_0|\mathbf{k}+\mathbf{q}|}{\Omega}\right) e^{il_1(\phi_0 \mp \theta_{\mathbf{k}+\mathbf{q}})}\right|^2 \\
\times\, \delta(k^2 - |\mathbf{k}+\mathbf{q}|^2 - \omega + l\Omega),
\end{aligned} \tag{H-12}$$

where $\mathbf{k} \to (k,\theta_k)$, $\mathbf{q} \to (q,\theta_q)$, and $|\mathbf{k}+\mathbf{q}| = \sqrt{k^2+q^2+2kq\cos(\theta_k-\theta_q)}$ are represented in polar coordinates. The superscript $\pm$ in $\bar{C}^{\pm}_{\rm 2D}(q,\theta_q;\omega)$ denotes left- or right-handed circularly polarized light. After carefully eliminating the Dirac delta function and integrating over θ_k, we obtain:

$$\begin{aligned}
\bar{C}^{\pm}_{\rm 2D}(q,\theta_q;\omega) = \frac{1}{(2\pi)^2}\sum_l \int_0^{+\infty} kdk \frac{f\left(\sqrt{k^2-\omega+l\Omega}\right)\bar{f}(k)}{\left|\sqrt{4k^2q^2-(q^2+\omega-l\Omega)^2}\right|}\left|J_l\left(\frac{2A_0}{\Omega}k\right)\right|^2 \\
\times\, 2\sum_{l_1}\sum_{l_2} J_{l+l_1}\left(\frac{2A_0}{\Omega}\sqrt{k^2-\omega+l\Omega}\right) J_{l+l_2}\left(\frac{2A_0}{\Omega}\sqrt{k^2-\omega+l\Omega}\right) \\
\times\, T_{(l_1-l_2)}\left(\frac{q^2-\omega+l\Omega}{2q\sqrt{k^2-\omega+l\Omega}}\right)\exp[+i(l_1-l_2)(\phi_0 \mp \theta_q)],
\end{aligned} \tag{H-13}$$

where $T_n(x)$ is the Chebyshev polynomial of the first kind. We observe that $\bar{C}^{\pm}_{\rm 2D}(q,\theta_q;\omega)$ depends on the angle θ_q, initial phase ϕ_0, and the helicity of the drive. However, integrating over θ_q leads to $\delta_{l_1 l_2}$ and yields a simpler final expression:

$$\overline{\bar{C}_{\rm 2D}(q,\omega)} \equiv \int_0^{2\pi} d\theta_q \bar{C}^{\pm}_{\rm 2D}(q,\theta_q;\omega) = \frac{2}{(2\pi)^2}\sum_l \int_0^{+\infty} kdk \frac{f\left(\sqrt{k^2-\omega+l\Omega}\right)\bar{f}(k)}{\left|\sqrt{4k^2q^2-(q^2+\omega-l\Omega)^2}\right|}\left|J_l\left(\frac{2A_0}{\Omega}k\right)\right|^2, \tag{H-14}$$

where we used the fact that $T_0(x) = 1$ and $\sum_l |J_l(x)|^2 = 1$.

Interestingly, despite the simplification achieved in Eq. (H-14), the underlying angular dependence of $\bar{C}^{\pm}_{\rm 2D}(q, \theta_q; \omega)$ reveals a subtle physical feature of the system. Even when the 2D system is driven by circularly polarized light, $\bar{C}(\mathbf{q}, \omega)$ is not fully isotropic in $\mathbf{q}$. This anisotropy manifests as an explicit dependence on the angle difference between the initial phase ϕ_0 of the drive and the direction of $\mathbf{q}$, represented by θ_q. This is because in the steady state the physical direction of flow of the fermion fluid rotates in time with the drive, so as to instantaneously remain orthogonal to the electric field. While invisible in the equal-time density correlations, this instantaneous directionality of the flow leads to an explicit angular dependence of the correlations measured at different times and different angles. But, despite the dependence of $\bar{C}(\mathbf{q}, \omega)$ on the direction of $\mathbf{q}$, its non-analyticities are located at frequencies that depend only on the magnitude of $|\mathbf{q}|$.

Appendix I: Persistence of non-analyticities in the presence of electron-electron interactions

In this section, we demonstrate that the non-analyticities in the Floquet occupation function arising from electron-phonon coupling persist even when electron-electron interactions are included in the Floquet-Boltzmann description. For the sake of brevity, we will use ϵ_p to denote Floquet energy in this section.

In the Floquet-Boltzmann framework, the steady state occupation function $f(\epsilon_p)$ is determined by the condition that the total collision integral vanishes:

$$I_{\rm tot}[f(\epsilon_p)] = I_{\text{e-ph}}[f(\epsilon_p)] + I_{\text{e-e}}[f(\epsilon_p)] = 0, \tag{I-1}$$

where $I_{\text{e-ph}}$ is the electron-phonon collision integral and $I_{\text{e-e}}$ is the electron-electron collision integral. As shown in the main text, $I_{\text{e-ph}}$ has the form [see Eq. (5) in the main text]:

$$I_{\text{e-ph}}[f(\epsilon_p)] = \sum_q \Big(f(\epsilon_q) W_{q\to p} \bar{f}(\epsilon_p) - f(\epsilon_p) W_{p\to q} \bar{f}(\epsilon_q) \Big), \tag{I-2}$$

with $\bar{f}(\epsilon_p) = 1 - f(\epsilon_p)$ and the scattering rate:

$$W_{q\to p} = \sum_l \Phi^{(l)}_{q,p} S(\epsilon_q - \epsilon_p + l\Omega). \tag{I-3}$$

As discussed in the main text and Appendix C "Analytical analysis of non-analyticities" of the supplementary material, the non-analyticities in the occupation function originate from non-analyticities in the S function. These non-analyticities appear at specific energies $\epsilon_* = n\Omega + s\omega_*$, where ω_* represents the energy at which the bath's density of states have a non-analyticity.

The electron-electron collision integral in the Floquet-Boltzmann framework has the general form (see e.g., Eq. (41) in Ref. [14]):

$$I_{\text{e-e}}[f(\epsilon_p)] = \sum_{k,q,q'} K(p,k,q,q') \left[f(\epsilon_p) f(\epsilon_k) \bar{f}(\epsilon_q) \bar{f}(\epsilon_{q'}) - \bar{f}(\epsilon_p) \bar{f}(\epsilon_k) f(\epsilon_q) f(\epsilon_{q'}) \right], \tag{I-4}$$

where $K(p,k,q,q')$ incorporates the Floquet interaction matrix elements, and enforces energy conservation modulo Ω to account for Floquet-Umklapp processes.

Let us consider the energy $\epsilon_p = \epsilon_*$ where non-analyticities occur in the absence of the the electron-electron collision operator. We will denote the n-th derivative of a function g with respect to ϵ_p used in Appendix C:

$$g^{[n]}(\epsilon_p) = \frac{d^n g(\epsilon_p)}{d\epsilon_p^n}. \tag{I-5}$$

In order to examine the presence of a non-analyticities, we define the following difference for any function F at a specific point ϵ_*:

$$\Delta F(\epsilon_*) = F(\epsilon_* + 0^+) - F(\epsilon_* - 0^-). \tag{I-6}$$

If the above limit is finite, then the function F has a discontinuity at ϵ_* from above versus from below.

We now differentiate Eq. (I-1) n times with respect to ϵ_p:

$$\frac{d^n}{d\epsilon_p^n} I_{\text{e-ph}}[f(\epsilon_p)] + \frac{d^n}{d\epsilon_p^n} I_{\text{e-e}}[f(\epsilon_p)] = 0. \tag{I-7}$$

And evaluate the limit of the above expression as defined in Eq. (I-6) around $\epsilon_p = \epsilon_*$:

$$\Delta\left(\frac{d^n}{d\epsilon_p^n} I_{\text{e-ph}}[f(\epsilon_p)]\right)_{\epsilon_p=\epsilon_*} + \Delta\left(\frac{d^n}{d\epsilon_p^n} I_{\text{e-e}}[f(\epsilon_p)]\right)_{\epsilon_p=\epsilon_*} = 0. \tag{I-8}$$

As shown in Appendix C, particularly in relation to Eqs. (C-16) and (C-23) [as well as Eqs. (D-5), (D-9), (D-12) for more specific models], the above term coming from electron-phonon collisions can be expressed as:

$$\Delta\left(\frac{d^n}{d\epsilon_p^n} I_{\text{e-ph}}[f(\epsilon_p)]\right)_{\epsilon_p=\epsilon_*} = A_{n,\epsilon_*}[f,S]\cdot\Delta f^{[n]}(\epsilon_*) + B_{n,\epsilon_*}[f,f^{[1]},\cdots,f^{[n-1]};S,S^{[1]},\cdots,S^{[n]}] \tag{I-9}$$

This decomposition has the following interpretation:

- $A_{n,\epsilon_*}[f,S]$ is a general way of writing the (functional) coefficient [see R_{ε_p,η_p} in Eq. (C-16)] that captures how changes in the n-th derivative of f affect the n-th derivative of the collision integral.
- $B_{n,\epsilon_*}[f,f^{[1]},\cdots,f^{[n-1]};S,S^{[1]},\cdots,S^{[n]}] \neq 0$ emerges from differentiating the S function in the collision integral, and corresponds to the right-hand side of Eq. (C-16). When the S function is differentiated n times, it can produce terms containing Dirac delta functions at $\epsilon_p = \epsilon_*$ and leads to $B_{n,\epsilon_*} \neq 0$ follwing the discussion in Appendix C 3.

Similarly, for the electron-electron collision integral, we can express the jump in its n-th derivative as:

$$\Delta\left(\frac{d^n}{d\epsilon_p^n} I_{\text{e-e}}[f(\epsilon_p)]\right)_{\epsilon_p=\epsilon_*} = C_{n,\epsilon_*}[f,K]\cdot\Delta f^{[n]}(\epsilon_*) + D_{n,\epsilon_*}[f,f^{[1]},\cdots,f^{[n-1]};K,K^{[1]},\cdots,K^{[n]}] \tag{I-10}$$

where C_{n,ϵ_*} and D_{n,ϵ_*} have similar interpretations to A_{n,ϵ_*} and B_{n,ϵ_*}, respectively, but arise from the structure of the electron-electron collision integral $I_{\text{e-e}}[f(\epsilon_p)]$.

Combining these equations, we get:

$$(A_{n,\epsilon_*} + C_{n,\epsilon_*})\cdot\Delta f^{[n]}(\epsilon_*) + (B_{n,\epsilon_*} + D_{n,\epsilon_*}) = 0 \tag{I-11}$$

The crucial point in our analysis is that generically $(B_{n,\epsilon_*} + D_{n,\epsilon_*}) \neq 0$ at the energy $\epsilon_p = \epsilon_*$. This is because:

- $B_{n,\epsilon_*} \neq 0$ as it originates from the non-analyticity in the S function at $\epsilon_p = \epsilon_*$ which contains the detailed information of the bosonic bath, such as its density of states.
- D_{n,ϵ_*} arises from a completely different physical origin, the electron-electron collision integral, and has no particular reason to exactly cancel B_{n,ϵ_*}. For D_{n,ϵ_*} to exactly cancel B_{n,ϵ_*}, the electron-electron interactions would need to be fine-tuned in a way that depends on the precise details of the bosonic bath, which is not physical.

Therefore, the equation $(A_{n,\epsilon_*} + C_{n,\epsilon_*})\cdot\Delta f^{[n]}(\epsilon_*) + (B_{n,\epsilon_*} + D_{n,\epsilon_*}) = 0$ implies $\Delta f^{[n]}(\epsilon_*) \neq 0$ [assuming $(A_{n,\epsilon_*} + C_{n,\epsilon_*})$ is finite, which is generally true]. This means that the non-analyticity in the n-th derivative of the occupation function persists even when electron-electron interactions are included in the Floquet-Boltzmann framework.

As a specific example, we can consider a 2D system coupled to a gapped bath with a gap at Δ. In this case, D_{n,ϵ_*} would be expected to be zero because the electron-electron collision integral does not contain any information about the bath. Consequently, the Floquet interaction kernel $K(p,k,q,q')$ is smooth around $\epsilon_* = \Delta$, which leads to $D_{n,\epsilon_*} = 0$. This simplifies Eq. (I-11) to:

$$\Delta f^{[n]}(\epsilon_*) = -\frac{B_{n,\epsilon_*}}{A_{n,\epsilon_*} + C_{n,\epsilon_*}}. \tag{I-12}$$

Therefore the non-analyticities in the steady state occupation function at energies ϵ_* persist even when electron-electron interactions are included as a collision integral. While the electron-electron collision integral may modify the magnitude of these non-analyticities, particularly as the relative strength of interactions changes, it cannot eliminate them entirely when their relative strength are finite, thus preserving the essential character of the Floquet non-Fermi liquid state.