## SUPPLEMENTARY MATERIAL FOR: "DISSIPATION-SHAPED QUANTUM GEOMETRY IN NONLINEAR TRANSPORT"

This Supplementary Material provides a detailed derivation of the second-order DC nonlinear conductivity $\sigma_{abc}$ for an electronic system coupled to a microscopic fermionic heat bath. We demonstrate the derivation of the non-equilibrium steady state (NESS) density matrix and its expansion to obtain the conductivity components presented in the main text.

### Appendix A: Microscopic Model and NESS Density Matrix

To establish a benchmark for the nonlinear conductivity, we derive the non-equilibrium steady state (NESS) density matrix from a microscopic open quantum system model. We consider a crystalline electronic system, $H_S(t)$, coupled to a non-interacting, wide-band fermionic bath, $H_B$, which models a metallic backgate.

The total Hamiltonian is $H(t) = H_S(t) + H_B + H_{SB}$, with components given by:

$$H(t) = \begin{bmatrix} H_S(t) & H_{SB} \\ H_{SB}^\dagger & H_B \end{bmatrix}. \tag{A-1}$$

The system Hamiltonian, $H_S(t) = H_0 + V(t)$, describes Bloch electrons ($H_0$) subject to a general time-dependent perturbation $V(t)$. $H_0 = \sum_n \epsilon_n |\chi_n\rangle\langle\chi_n|$, where $|\chi_n\rangle$ and $\epsilon_n$ are the unperturbed eigenstates and energies, respectively. The indices $m, n, l$ hereafter are general, incorporating momentum, band, and spin degrees of freedom. This model is in the same class of those non-interacting fermionic models often described within the Keldysh formalism [26, 59–64, 66].

The environment is modeled as a featureless fermionic bath, $H_B = \sum_{n,i} \varepsilon_i |\varphi_{n,i}\rangle\langle\varphi_{n,i}|$, where each system state $|\chi_n\rangle$ couples independently to a set of bath states $|\varphi_{n,i}\rangle$. The tunnel coupling is $H_{SB} = \lambda \sum_{n,i}(|\chi_n\rangle\langle\varphi_{n,i}| + \text{h.c.})$. We assume the bath is initially in thermal equilibrium, described by the Fermi-Dirac distribution $f_0(\epsilon)$ at chemical potential $\mu$ and temperature $T = 1/\beta$.

Tracing out the bath degrees of freedom under the wide-band approximation (constant bath density of states, $\nu_B(\omega) = \nu_0$) yields the dynamics for the system's reduced density matrix $\rho_S(t)$. This procedure defines the physical relaxation rate $\Gamma = \lambda^2\nu_0/2$, which microscopically regularizes the system dynamics.

We solve for the NESS density matrix by expanding $\rho_S(t)$ perturbatively in the driving perturbation $V(t)$:

$$\rho_S(t) = \rho^{(0)} + \rho^{(1)}(t) + \rho^{(2)}(t) + \mathcal{O}(V^3). \tag{A-2}$$

The zeroth-order term, or equilibrium NESS, is diagonal:

$$\rho^{(0)}_{mn} = \delta_{mn} \int_{-\infty}^{\infty} \frac{d\omega_b}{2\pi} \frac{2\Gamma}{\omega_b^2 + \Gamma^2} f_0(\epsilon_m + \omega_b) \equiv \delta_{mn} \tilde{\rho}^{(0)}_n. \tag{A-3}$$

This expression represents the equilibrium Fermi-Dirac distribution, broadened by the system-bath coupling $\Gamma$. In the ideal-bath limit ($\Gamma \to 0$), $\tilde{\rho}^{(0)}_n \to f_0(\epsilon_n)$.

The first- and second-order corrections to the density matrix define the response kernels. We express them in the frequency domain, where $V(t) = \int V(\omega) e^{-i\omega t} d\omega/(2\pi)$ and $\rho^{(n)}(t) = \int \rho^{(n)}(\omega) e^{-i\omega t} d\omega/(2\pi)$. The response kernels $\tilde{\rho}^{(1)}$ and $\tilde{\rho}^{(2)}$ are found to be:

$$\rho^{(1)}_{mn}(\omega) = V_{mn}(\omega)\, \tilde{\rho}^{(1)}_{mn}(\omega), \quad \tilde{\rho}^{(1)}_{mn}(\omega) = \int_{-\infty}^{\infty} \frac{d\omega_b}{2\pi} \frac{2\Gamma}{\omega_b^2 + \Gamma^2} \frac{f_0(\epsilon_n + \omega_b) - f_0(\epsilon_m - \omega_b)}{\omega + \omega_b + \epsilon_{nm} + i\Gamma}, \tag{A-4}$$

and

$$\begin{aligned} \rho^{(2)}_{mn}(\omega_1 + \omega_2) &= \sum_l V_{ml}(\omega_1) V_{ln}(\omega_2) \tilde{\rho}^{(2)}_{mln}(\omega_1, \omega_2), \\ \tilde{\rho}^{(2)}_{mln}(\omega_1, \omega_2) &= \int_{-\infty}^{\infty} \frac{d\omega_b}{2\pi} \frac{2\Gamma}{\omega_b^2 + \Gamma^2} \Bigg[ \frac{f_0(\epsilon_m - \omega_b)}{(\omega_1 + \omega_2 + \omega_b + \epsilon_{nm} + i\Gamma)(\omega_1 + \omega_b + \epsilon_{lm} + i\Gamma)} \\ &+ \frac{f_0(\epsilon_n + \omega_b)}{(\omega_1 + \omega_2 + \omega_b + \epsilon_{nm} + i\Gamma)(\omega_2 + \omega_b + \epsilon_{nl} + i\Gamma)} - \frac{f_0(\epsilon_l + \omega_b)}{(\omega_1 + \omega_b + \epsilon_{lm} + i\Gamma)(\omega_2 - \omega_b + \epsilon_{nl} + i\Gamma)} \Bigg], \end{aligned} \tag{A-5}$$

where $\epsilon_{nm} \equiv \epsilon_n - \epsilon_m$. These expressions, derived directly from the open-system dynamics, provide the exact response kernels for a system coupled to a fermionic bath. They differ from phenomenological models (e.g., relaxation time approximations and imaginary frequency regularizations) and form the basis for calculating the nonlinear conductivity.

### Alternative Representation

The integrals over the bath variable $\omega_b$ in Eqs. (A-3) to (A-5) can be performed using the Cauchy residue theorem. This leads to an alternative representation in terms of the Polygamma function $\psi^{(0)}$. We find the broadened zero-th kernel

$$\rho^{(0)}_{mn} = \delta_{mn}\frac{1}{2}\Big[f_+(\epsilon_n) + f_-(\epsilon_n)\Big] \equiv \tilde{\rho}^{(0)}_n, \quad f_\pm(\epsilon) = \frac{1}{2} \pm \frac{i}{\pi}\psi^{(0)}\left(\frac{1}{2} \pm i\beta\frac{\epsilon \mp i\Gamma - \mu}{2\pi}\right), \tag{A-6}$$

where $\psi^{(0)}$ is the 0-th order Polygamma function. Importantly,

$$\lim_{\Gamma\to 0}\tilde{\rho}^{(0)}_n = \frac{1}{1+\exp[\beta(\epsilon_n - \mu)]}, \tag{A-7}$$

which shows that $\tilde{\rho}^{(0)}_n$ reduces to the ideal Fermi-Dirac distribution in the limit of $\Gamma \to 0$. Moreover, for the first- and second-order expansions, we have

$$\tilde{\rho}^{(1)}_{mn}(\omega) = \frac{\tilde{\rho}^{(0)}_n - \tilde{\rho}^{(0)}_m}{\omega + \epsilon_{nm} + 2i\Gamma} + \frac{r^{(1)}_+(\omega, \epsilon_m, \epsilon_n) + r^{(1)}_-(\omega, \epsilon_m, \epsilon_n)}{\omega + \epsilon_{nm} + 2i\Gamma}, \tag{A-8}$$

with

$$r^{(1)}_+(\omega, \epsilon_m, \epsilon_n) = i\Gamma\frac{f_+(\epsilon_n) - f_+(\epsilon_m - \omega)}{\omega + \epsilon_{nm}}, \quad r^{(1)}_-(\omega, \epsilon_m, \epsilon_n) = i\Gamma\frac{f_-(\epsilon_n + \omega) - f_-(\epsilon_m)}{\omega + \epsilon_{nm}}, \tag{A-9}$$

and

$$\tilde{\rho}^{(2)}_{mln}(\omega_1, \omega_2) = \frac{\tilde{\rho}^{(1)}_{ln}(\omega_2) - \tilde{\rho}^{(1)}_{ml}(\omega_1)}{\omega_1 + \omega_2 + \epsilon_{nm} + 2i\Gamma} + \frac{r^{(2)}_+(\omega_1, \omega_2, \epsilon_m, \epsilon_l, \epsilon_n) + r^{(2)}_-(\omega_1, \omega_2, \epsilon_m, \epsilon_l, \epsilon_n)}{\omega_1 + \omega_2 + \epsilon_{nm} + 2i\Gamma}, \tag{A-10}$$

in which

$$r^{(2)}_+(\omega_1, \omega_2, \epsilon_m, \epsilon_l, \epsilon_n) = \frac{r^{(1)}_+(\omega_2, \epsilon_l, \epsilon_n) - r^{(1)}_+(\omega_1, \epsilon_m - \omega_2, \epsilon_l - \omega_2)}{\omega_1 + \omega_2 + \epsilon_{nm}}, \tag{A-11}$$

$$r^{(2)}_-(\omega_1, \omega_2, \epsilon_m, \epsilon_l, \epsilon_n) = \frac{r^{(1)}_-(\omega_2, \epsilon_l + \omega_1, \epsilon_n + \omega_1) - r^{(1)}_-(\omega_1, \epsilon_m, \epsilon_l)}{\omega_1 + \omega_2 + \epsilon_{nm}}. \tag{A-12}$$

From Eqs. (A-8) and (A-12), it is clear that the perturbative expansions of the density matrix for the system coupled with the featureless fermionic bath, obtained by solving the open system Schrödinger equation exactly, have striking differences with those obtained from conventional perturbation theories assuming an adiabatic turning-on (IFRs) or a hand-given relaxation (RTAs).

## Appendix B: General Expression for Nonlinear Conductivity

We now apply the microscopic NESS formalism to derive the general expression for the second-order nonlinear conductivity, $\sigma_{abc}$. The physical perturbation $V(t)$ is induced by a uniform electric field $\mathbf{E}(t)$ via the Peierls substitution, $H_S(t) = H_0(\mathbf{k} - \mathbf{A}(t))$, where $\mathbf{E}(t) = -\partial_t\mathbf{A}(t)$.

The perturbation $V(t) = H_S(t) - H_0$ and the current operator $J_a(t) = \partial H_S(t)/\partial k_a$ are expanded in powers of the vector potential $\mathbf{A}(t)$. Using $\partial_a \equiv \partial_{k_a}$, we have:

$$V(t) = V^{(1)}(t) + V^{(2)}(t) + \mathcal{O}(A^3) = \sum_b(-\partial_b H_0)A_b(t) + \sum_{b,c}\frac{1}{2}(\partial_b\partial_c H_0)A_b(t)A_c(t) + \dots \tag{B-1}$$

$$J_a(t) = J^{(0)}_a + J^{(1)}_a(t) + J^{(2)}_a(t) + \mathcal{O}(A^3) = (\partial_a H_0) + \sum_c(-\partial_a\partial_c H_0)A_c(t) + \sum_{b,c}\frac{1}{2}(\partial_a\partial_b\partial_c H_0)A_b(t)A_c(t) + \dots \tag{B-2}$$

The total nonlinear current is the expectation value $j_a(t) = \mathrm{Tr}[\rho_S(t) J_a(t)]$. The component at second order in the field, $j_a^{(2)}(t)$, receives contributions from all combinations of density matrix and current operator orders that multiply to $\mathcal{O}(A^2)$:

$$j_a^{(2)}(t) = \mathrm{Tr}\big[\rho^{(0)} J_a^{(2)}(t)\big] + \mathrm{Tr}\big[\rho^{(1)}(t) J_a^{(1)}(t)\big] + \mathrm{Tr}\big[\rho^{(2)}(t) J_a^{(0)}\big]. \tag{B-3}$$

It is important to correctly identify the sources for the density matrix terms:

- $\rho^{(1)}(t)$ contains terms linear in $V$, so it has parts from $V^{(1)}$ and $V^{(2)}$: $\rho^{(1)}(t) = \rho^{(1)}[V^{(1)}] + \rho^{(1)}[V^{(2)}] + \ldots$
- $\rho^{(2)}(t)$ contains terms quadratic in $V$. The $\mathcal{O}(A^2)$ term arises from $V^{(1)}V^{(1)}$: $\rho^{(2)}(t) = \rho^{(2)}[V^{(1)}, V^{(1)}] + \ldots$

This leads to four distinct contributions to the second-order current:

$$j_a^{(2)}(t) = \underbrace{\mathrm{Tr}[\rho^{(0)} J_a^{(2)}(t)]}_{\text{(I)}} + \underbrace{\mathrm{Tr}[\rho^{(1)}[V^{(1)}] J_a^{(1)}(t)]}_{\text{(II)}} + \underbrace{\mathrm{Tr}[\rho^{(1)}[V^{(2)}] J_a^{(0)}]}_{\text{(III)}} + \underbrace{\mathrm{Tr}[\rho^{(2)}[V^{(1)}, V^{(1)}] J_a^{(0)}]}_{\text{(IV)}}. \tag{B-4}$$

The conductivity tensor $\sigma_{abc}(\omega_1, \omega_2)$ is defined in frequency space via $j_a^{(2)}(\omega_1+\omega_2) = \sigma_{abc}(\omega_1,\omega_2) E_b(\omega_1) E_c(\omega_2)$. Using $A_b(\omega) = E_b(\omega)/(i\omega)$ and the response kernels from Eqs. (A-4) to (A-5), we map each of the four terms to the conductivity.

We introduce the operator matrix elements for the Fourier components, corresponding to the notation in: $V_{mn}^{(b)} \equiv \langle m|(-\partial_b H_0)|n\rangle$, $V_{mn}^{(bc)} \equiv \langle m|(\partial_b\partial_c H_0)|n\rangle/2$, $J_{mn}^{(a)} \equiv \langle m|(\partial_a H_0)|n\rangle = -V_{mn}^{(a)}$, $J_{mn}^{(ac)} \equiv \langle m|(-\partial_a\partial_c H_0)|n\rangle = -V_{mn}^{(ac)}$, and $J_{mn}^{(abc)} \equiv \langle m|\partial_a\partial_b\partial_c H_0|n\rangle/2$. Translating terms (I)-(IV) into the full frequency-dependent conductivity, we obtain the expression:

$$\sigma_{abc}(\omega_1,\omega_2) = \frac{i}{\omega_1}\frac{i}{\omega_2}\sum_{m,l,n}\Bigg[\underbrace{V_{ml}^{(b)} V_{ln}^{(c)} \tilde{\rho}_{mln}^{(2)}(\omega_1,\omega_2) J_{nm}^{(a)}}_{\text{from (IV): } \rho^{(2)}[V^{(1)},V^{(1)}]J^{(0)}} + \underbrace{V_{mn}^{(bc)} \tilde{\rho}_{mn}^{(1)}(\omega_1+\omega_2) J_{nm}^{(a)}}_{\text{from (III): } \rho^{(1)}[V^{(2)}]J^{(0)}}$$
$$+ \underbrace{V_{mn}^{(b)} \tilde{\rho}_{mn}^{(1)}(\omega_1) J_{nm}^{(ac)}}_{\text{from (II): } \rho^{(1)}[V^{(1)}]J^{(1)}} + \underbrace{\rho_{mn}^{(0)} J_{nm}^{(abc)}}_{\text{from (I): } \rho^{(0)}J^{(2)}}\Bigg] + \begin{pmatrix} b \leftrightarrow c \\ \omega_1 \leftrightarrow \omega_2 \end{pmatrix}. \tag{B-5}$$

This expression is the exact second-order conductivity for the microscopic model. The first term, $\rho_{mn}^{(0)} J_{nm}^{(abc)}$, is the $\mathcal{O}(A^2)$ correction to the current operator itself (term I). The second term, involving $J_{a,nm}^{(c)}$, arises from the $\mathcal{O}(A)$ correction to the density matrix and the $\mathcal{O}(A)$ correction to the current operator (term II). The final two terms, involving $J_{nm}^{(a)}$, arise from the second-order correction to the density matrix, which has a component from the first-order response to the $\mathcal{O}(A^2)$ perturbation (term III, $V^{(2)}$) and a component from the second-order response to the $\mathcal{O}(A)$ perturbation (term IV, $V^{(1)}V^{(1)}$).

The DC conductivity,

$$\sigma_{abc} \equiv \lim_{\omega_1\to 0}\Big[\lim_{\omega_2\to-\omega_1} \sigma_{abc}(\omega_1,\omega_2)\Big], \tag{B-6}$$

is obtained by taking two sequential limits of Eq. (B-5). The specific structure of the NESS kernels $\tilde{\rho}^{(n)}$, particularly their regularization by $\Gamma$, decides the final form of the intrinsic ($\Gamma$-independent) and extrinsic ($\Gamma$-dependent) contributions.

## Appendix C: Structure of Second-Order Expansion Coefficients

In this section, we provide the detailed analytical expressions for the second-order expansion coefficients of the density matrix response kernels, $\tilde{\rho}_{mn}^{(1)}(\omega)$ and $\tilde{\rho}_{mln}^{(2)}(\omega, \omega_2 \to -\omega)$, derived from the microscopic open-system formalism for a generic multiband model. These coefficients are essential for the regularization of the DC limit and the calculation of the nonlinear conductivity.

We define the expansion coefficients $\mathcal{C}_{mn}^{(1,k)}$ and $\mathcal{C}_{mln}^{(2,k)}$ (for the specific frequency configuration $\omega_1 = \omega, \omega_2 = -\omega$) via Taylor expansion around $\omega = 0$:

$$\tilde{\rho}_{mn}^{(1)}(\omega) = \sum_{k=0}^{\infty} \mathcal{C}_{mn}^{(1,k)}\,\omega^k, \quad \tilde{\rho}_{mln}^{(2)}(\omega,-\omega) = \sum_{k=0}^{\infty} \mathcal{C}_{mln}^{(2,k)}\,\omega^k. \tag{C-1}$$

We verified analytically that the current $\lim_{\omega'\to-\omega} j_a^{(2)}(\omega+\omega')$ from the zeroth-order and first-order coefficients ($k=0,1$) are exactly zero, and present the results for the second-order coefficients ($k=2$). The expressions are given in terms of the Polygamma functions $\psi^{(n)}(z)$. We utilize the shorthand notation $\epsilon_{nm}=\epsilon_n-\epsilon_m$ for the energy difference and define the arguments of the Polygamma functions as:

$$z_{n,\pm} = \frac{1}{2} + \frac{\beta}{2\pi}(\Gamma \pm i(\epsilon_n - \mu)) \quad z^0_{n,\pm} = \frac{1}{2} \pm \frac{i\beta}{2\pi}(\epsilon_n - \mu) \tag{C-2}$$

where $\beta$ is the inverse temperature, $\Gamma$ is the bath-induced relaxation rate, and $\mu$ is the chemical potential.

### 1. First-Order Kernel Coefficients

#### a. *Intraband Coefficient:* $\mathcal{C}^{(1,2)}_{nn}$

$$\begin{aligned}\mathcal{C}^{(1,2)}_{nn} = \frac{\beta}{96\pi^4\Gamma^2}\Big[&6\pi^2\left(\psi^{(1)}(z_{n,+})+\psi^{(1)}(z_{n,-})\right) - 3\pi\beta\Gamma\left(\psi^{(2)}(z_{n,+})+\psi^{(2)}(z_{n,-})\right)\\ &+(\beta\Gamma)^2\left(\psi^{(3)}(z_{n,+})+\psi^{(3)}(z_{n,-})\right)\Big]\end{aligned} \tag{C-3}$$

Its Taylor expansion in $\Gamma$ is:

$$\mathcal{C}^{(1,2)}_{nn} = \frac{1}{\Gamma^2}\left[\frac{\beta}{16\pi^2}\sum_{\pm}\psi^{(1)}(z^0_{n,\pm})\right] + \left[\frac{\beta^3}{384\pi^4}\sum_{\pm}\psi^{(3)}(z^0_{n,\pm})\right] + \mathcal{O}(\Gamma) \tag{C-4}$$

where $\psi^{(n)}$ is the $n$-th order Polygamma function, which can be converted to $n$-th order derivative over the Fermi-Dirac function [see Eq. (A-6)].

#### b. *Interband Coefficient:* $\mathcal{C}^{(1,2)}_{nm}$

$$\begin{aligned}\mathcal{C}^{(1,2)}_{nm} = \frac{i}{8\pi^3\epsilon^3_{nm}(2i\Gamma-\epsilon_{nm})^2}\Big[&4\pi^2(2\Gamma+i\epsilon_{nm})^2\left(-\psi^{(0)}(z_{n,+})+\psi^{(0)}(z_{m,+})+\psi^{(0)}(z_{n,-})-\psi^{(0)}(z_{m,-})\right)\\ &+8\pi\beta\Gamma\epsilon_{nm}(i\Gamma-\epsilon_{nm})\left(\psi^{(1)}(z_{n,+})+\psi^{(1)}(z_{m,-})\right)\\ &+\beta^2\Gamma\epsilon^2_{nm}(2\Gamma+i\epsilon_{nm})\left(\psi^{(2)}(z_{n,+})+\psi^{(2)}(z_{m,-})\right)\Big]\end{aligned} \tag{C-5}$$

Its Taylor expansion in $\Gamma$ is:

$$\mathcal{C}^{(1,2)}_{nm} = -\frac{i}{2\pi\epsilon^3_{nm}}\left[\psi^{(0)}(z^0_{n,-})-\psi^{(0)}(z^0_{n,+})-\psi^{(0)}(z^0_{m,-})+\psi^{(0)}(z^0_{m,+})\right]+\mathcal{O}(\Gamma) \tag{C-6}$$

### 2. Second-Order Kernel Coefficients

#### a. *Fully Intraband Coefficient:* $\mathcal{C}^{(2,2)}_{nnn}$

$$\mathcal{C}^{(2,2)}_{nnn} = \frac{i\beta^4}{768\pi^5}\left[\psi^{(4)}(z_{n,+})-\psi^{(4)}(z_{n,-})\right] \tag{C-7}$$

Its Taylor expansion in Γ is:

$$\mathcal{C}^{(2,2)}_{nnn} = \frac{i\beta^4}{768\pi^5}\left[\psi^{(4)}(z^0_{n,+}) - \psi^{(4)}(z^0_{n,-})\right] + \mathcal{O}(\Gamma) \tag{C-8}$$

The presence of the fourth-order Polygamma function $\psi^{(4)}$ is a direct consequence of the regularization imposed by the fermionic bath. This specific analytical structure is responsible for the emergence of the intrinsic kinetic contribution $\sigma^{\mathrm{kin}}_{abc} \propto f^{(4)}_0$ in the clean limit (see Section D 4).

### b. Mixed Coefficient: $\mathcal{C}^{(2,2)}_{nmn}$

The expression for $\mathcal{C}^{(2,2)}_{nmn}$ is composed of terms involving $\psi^{(0)}$, $\psi^{(1)}$, and $\psi^{(2)}$:

$$\mathcal{C}^{(2,2)}_{nmn} = P_0 + P_1 + P_2 \tag{C-9}$$

where

$$P_0 = -\frac{6i\pi}{4\pi^2\epsilon^4_{nm}}\left[\psi^{(0)}(z_{n,+}) - \psi^{(0)}(z_{m,+}) - \psi^{(0)}(z_{n,-}) + \psi^{(0)}(z_{m,-})\right] \tag{C-10}$$

$$\begin{aligned} P_1 = \frac{\beta\epsilon_{nm}}{(4\Gamma^2+\epsilon^2_{nm})^2}\Big[&(4\Gamma^2 - 2i\Gamma\epsilon_{nm} + \epsilon^2_{nm})(2i\Gamma - \epsilon_{nm})^2\psi^{(1)}(z_{n,-}) \\ &+ (4\Gamma^2 + 2i\Gamma\epsilon_{nm} + \epsilon^2_{nm})(2i\Gamma + \epsilon_{nm})^2\psi^{(1)}(z_{n,+}) \\ &+ 2\Gamma(4\Gamma + 3i\epsilon_{nm})(2i\Gamma + \epsilon_{nm})^2\psi^{(1)}(z_{m,-}) \\ &- 2i\Gamma(4i\Gamma + 3\epsilon_{nm})(2i\Gamma - \epsilon_{nm})^2\psi^{(1)}(z_{m,+})\Big] \end{aligned} \tag{C-11}$$

and

$$\begin{aligned} P_2 = -\frac{i\beta^2\epsilon^2_{nm}}{4\pi(4\Gamma^2+\epsilon^2_{nm})}\Big[&\epsilon_{nm}(2i\Gamma + \epsilon_{nm})\psi^{(2)}(z_{n,+}) + 2\Gamma(2\Gamma + i\epsilon_{nm})\psi^{(2)}(z_{m,+}) \\ &+ \epsilon_{nm}(2i\Gamma - \epsilon_{nm})\psi^{(2)}(z_{n,-}) - 2\Gamma(2\Gamma - i\epsilon_{nm})\psi^{(2)}(z_{m,-})\Big] \end{aligned} \tag{C-12}$$

Its Taylor expansion in Γ is:

$$\begin{aligned} \mathcal{C}^{(2,2)}_{nmn} = \Bigg[&\frac{3i}{2\pi\epsilon^4_{nm}}\left(\psi^{(0)}(z^0_{n,-}) - \psi^{(0)}(z^0_{n,+}) - \psi^{(0)}(z^0_{m,-}) + \psi^{(0)}(z^0_{m,+})\right) \\ &+ \frac{\beta}{4\pi^2\epsilon^3_{nm}}\sum_{\pm}\psi^{(1)}(z^0_{n,\pm}) - \frac{i\beta^2}{16\pi^3\epsilon^2_{nm}}\left(\psi^{(2)}(z^0_{n,+}) - \psi^{(2)}(z^0_{n,-})\right)\Bigg] + \mathcal{O}(\Gamma) \end{aligned} \tag{C-13}$$

### c. Mixed Coefficient: $\mathcal{C}^{(2,2)}_{nnm}$

The expression for $\mathcal{C}^{(2,2)}_{nnm}$ involves terms up to $\psi^{(3)}$.

$$\mathcal{C}^{(2,2)}_{nnm} = \frac{1}{96\pi^4(2i\Gamma - \epsilon_{nm})}\left[Q_0 + Q_1 + Q_2 + Q_3\right] \tag{C-14}$$

where the terms $Q_n$ correspond to contributions involving $\psi^{(n)}$:

$$\begin{aligned} Q_0 = \frac{48\pi^3}{\epsilon^4_{nm}}\Big[&-i(-2i\Gamma + \epsilon_{nm})\psi^{(0)}(z_{n,-}) + (2\Gamma + i\epsilon_{nm})\psi^{(0)}(z_{n,+}) \\ &+ (2\Gamma + i\epsilon_{nm})\psi^{(0)}(z_{m,-}) - i(-2i\Gamma + \epsilon_{nm})\psi^{(0)}(z_{m,+})\Big] \end{aligned} \tag{C-15}$$

$$Q_1 = 6\pi^2\beta\left[\frac{1}{\Gamma^2}\left(-1-\frac{8i\Gamma^3}{\epsilon_{nm}^3}\right)\psi^{(1)}(z_{n,-}) - \frac{16\Gamma(\Gamma+i\epsilon_{nm})}{\epsilon_{nm}^2(2i\Gamma-\epsilon_{nm})^2}\psi^{(1)}(z_{m,-}) + \left(-\frac{1}{\Gamma^2}+\frac{4}{\epsilon_{nm}^2}-\frac{8i\Gamma}{\epsilon_{nm}^3}-\frac{4}{(2i\Gamma-\epsilon_{nm})^2}\right)\psi^{(1)}(z_{n,+})\right] \tag{C-16}$$

$$Q_2 = 3\pi\beta^2\left[\frac{4\Gamma^2+\epsilon_{nm}^2}{\Gamma\epsilon_{nm}^2}\psi^{(2)}(z_{n,-}) + \frac{1}{\Gamma}\left(1+\frac{8\Gamma^2(i\Gamma-\epsilon_{nm})}{\epsilon_{nm}^2(-2i\Gamma+\epsilon_{nm})}\right)\psi^{(2)}(z_{n,+}) + \frac{4\Gamma}{\epsilon_{nm}(2i\Gamma-\epsilon_{nm})}\psi^{(2)}(z_{m,-})\right] \tag{C-17}$$

$$Q_3 = \frac{\beta^3(2i\Gamma-\epsilon_{nm})}{\epsilon_{nm}}\left[\psi^{(3)}(z_{n,-})+\psi^{(3)}(z_{n,+})\right] \tag{C-18}$$

Its Taylor expansion in $\Gamma$ is:

$$\begin{aligned}\mathcal{C}_{nnm}^{(2,2)} =& \frac{1}{\Gamma^2}\left[\frac{\beta}{16\pi^2\epsilon_{nm}}\sum_{\pm}\psi^{(1)}(z_{n,\pm}^0)\right] + \frac{1}{\Gamma}\left[\frac{i\beta}{8\pi^2\epsilon_{nm}^2}\sum_{\pm}\psi^{(1)}(z_{n,\pm}^0)\right] \\ &+ \Bigg[-\frac{\beta}{4\pi^2\epsilon_{nm}^3}\sum_{\pm}\psi^{(1)}(z_{n,\pm}^0) + \frac{\beta^3}{384\pi^4\epsilon_{nm}}\sum_{\pm}\psi^{(3)}(z_{n,\pm}^0) \\ &+ \frac{i}{2\pi\epsilon_{nm}^4}\left(\psi^{(0)}(z_{n,-}^0)-\psi^{(0)}(z_{n,+}^0)-\psi^{(0)}(z_{m,-}^0)+\psi^{(0)}(z_{m,+}^0)\right)\Bigg] + \mathcal{O}(\Gamma)\end{aligned} \tag{C-19}$$

*d. Mixed Coefficient:* $\mathcal{C}_{nmm}^{(2,2)}$

Similarly, the expression for $\mathcal{C}_{nmm}^{(2,2)}$ is given by:

$$\mathcal{C}_{nmm}^{(2,2)} = \frac{1}{96\pi^4(2i\Gamma-\epsilon_{nm})}\left[U_0+U_1+U_2+U_3\right] \tag{C-20}$$

where

$$\begin{aligned}U_0 = \frac{48\pi^3}{\epsilon_{nm}^4}\Big[&(2\Gamma+i\epsilon_{nm})\psi^{(0)}(z_{n,-}) - i(-2i\Gamma+\epsilon_{nm})\psi^{(0)}(z_{m,+}) \\ &- i(-2i\Gamma+\epsilon_{nm})\psi^{(0)}(z_{m,-}) + (2\Gamma+i\epsilon_{nm})\psi^{(0)}(z_{m,+})\Big]\end{aligned} \tag{C-21}$$

$$\begin{aligned}U_1 = 6\pi^2\beta\Bigg[&\frac{16\Gamma(\Gamma+i\epsilon_{nm})}{\epsilon_{nm}^2(2i\Gamma-\epsilon_{nm})^2}\psi^{(1)}(z_{n,+}) + \frac{1}{\Gamma^2}\left(1+\frac{8i\Gamma^3}{\epsilon_{nm}^3}\right)\psi^{(1)}(z_{m,+}) \\ &+ \frac{1}{\Gamma^2}\left(1+4\Gamma^2\left(\frac{1}{(2i\Gamma-\epsilon_{nm})^2}+\frac{2i\Gamma-\epsilon_{nm}}{\epsilon_{nm}^3}\right)\right)\psi^{(1)}(z_{m,-})\Bigg]\end{aligned} \tag{C-22}$$

$$\begin{aligned}U_2 = 3\pi\beta^2\Bigg[&\frac{4\Gamma}{\epsilon_{nm}(-2i\Gamma+\epsilon_{nm})}\psi^{(2)}(z_{n,+}) - \frac{4\Gamma^2+\epsilon_{nm}^2}{\Gamma\epsilon_{nm}^2}\psi^{(2)}(z_{m,+}) \\ &+ \frac{1}{\Gamma}\left(-1+\frac{4\Gamma^2}{\epsilon_{nm}^2}\left(2-\frac{2i\Gamma}{2i\Gamma-\epsilon_{nm}}\right)\right)\psi^{(2)}(z_{m,-})\Bigg]\end{aligned} \tag{C-23}$$

$$U_3 = \frac{\beta^3(-2i\Gamma+\epsilon_{nm})}{\epsilon_{nm}}\left[\psi^{(3)}(z_{m,-})+\psi^{(3)}(z_{m,+})\right] \tag{C-24}$$

Its Taylor expansion in $\Gamma$ is:

$$\begin{aligned}\mathcal{C}^{(2,2)}_{nmm} =&\frac{1}{\Gamma^2}\left[-\frac{\beta}{16\pi^2\epsilon_{nm}}\sum_{\pm}\psi^{(1)}(z^0_{m,\pm})\right]+\frac{1}{\Gamma}\left[-\frac{i\beta}{8\pi^2\epsilon^2_{nm}}\sum_{\pm}\psi^{(1)}(z^0_{m,\pm})\right]\\ &+\left[\frac{\beta}{4\pi^2\epsilon^3_{nm}}\sum_{\pm}\psi^{(1)}(z^0_{m,\pm})-\frac{\beta^3}{384\pi^4\epsilon_{nm}}\sum_{\pm}\psi^{(3)}(z^0_{m,\pm})\right.\\ &\left.-\frac{i}{2\pi\epsilon^4_{nm}}\left(\psi^{(0)}(z^0_{n,-})-\psi^{(0)}(z^0_{n,+})-\psi^{(0)}(z^0_{m,-})+\psi^{(0)}(z^0_{m,+})\right)\right]+\mathcal{O}(\Gamma)\end{aligned} \quad \text{(C-25)}$$

These expressions highlight the complex analytical structure of the response kernels, characterized by the interplay between the energy denominators ($\epsilon_{nm}$) and the relaxation rate ($\Gamma$), reflecting the specific regularization imposed by the fermionic bath.

### e. *Fully Interband Coefficient:* $\mathcal{C}^{(2,2)}_{nml}$

The fully interband coefficient involves three distinct band indices, $n, m$, and $l$, and comprises contributions from $\psi^{(0)}$, $\psi^{(1)}$, and $\psi^{(2)}$:

$$\mathcal{C}^{(2,2)}_{nml} = \frac{1}{2i\Gamma-\epsilon_{nm}}(R_0+R_1+R_2) \quad \text{(C-26)}$$

where

$$\begin{aligned}R_0 =&\psi^{(0)}(z_{n-})\frac{2\Gamma+i\epsilon_{nm}}{2\pi\epsilon_{nm}\epsilon^3_{nl}}-\frac{i\psi^{(0)}(z_{n+})(-2i\Gamma+\epsilon_{nm})}{2\pi\epsilon_{nm}\epsilon^3_{nl}}+\psi^{(0)}(z_{m+})\frac{2\Gamma+i\epsilon_{nm}}{2\pi\epsilon_{nm}\epsilon^3_{ml}}-\frac{i\psi^{(0)}(z_{m-})(-2i\Gamma+\epsilon_{nm})}{2\pi\epsilon_{nm}\epsilon^3_{ml}}\\ &-\frac{\psi^{(0)}(z_{l-})(2\Gamma+i\epsilon_{nm})(\epsilon^2_{nm}+3\epsilon_{nl}\epsilon_{ml})}{2\pi\epsilon^2_{nl}\epsilon^2_{ml}}+\frac{\psi^{(0)}(z_{l+})(2\Gamma+i\epsilon_{nm})(\epsilon^2_{nm}+3\epsilon_{nl}\epsilon_{ml})}{2\pi\epsilon^2_{nl}\epsilon^2_{ml}}\end{aligned} \quad \text{(C-27)}$$

$$\begin{aligned}R_1 =&-\beta\Gamma\psi^{(1)}(z_{m-})\frac{(\Gamma-i\epsilon_{ml})}{\pi^2\epsilon^2_{ml}(2i\Gamma+\epsilon_{ml})^2}+\beta\Gamma\psi^{(1)}(z_{n+})\frac{\Gamma+i\epsilon_{nl}}{\pi^2\epsilon^2_{nl}(2i\Gamma-\epsilon_{nl})^2}\\ &+\frac{i\beta\Gamma\psi^{(1)}(z_{l+})\left(4\Gamma^2(\epsilon_{nl}+\epsilon_{ml})+\epsilon_{nm}(2\epsilon_{nl}+\epsilon_{ml})\epsilon_{ml}+2i\Gamma(\epsilon^2_{nm}-3\epsilon^2_{ml})\right)}{2\pi^2\epsilon^2_{nl}\epsilon^2_{ml}(2i\Gamma+\epsilon_{ml})^2}\\ &+\frac{i\beta\Gamma\psi^{(1)}(z_{l-})\left(4\Gamma^2(\epsilon_{nl}+\epsilon_{ml})-\epsilon_{nm}(\epsilon_{nl}+2\epsilon_{ml})\epsilon_{nl}+2i\Gamma(-\epsilon^2_{nm}+3\epsilon^2_{nl})\right)}{2\pi^2\epsilon^2_{nl}\epsilon^2_{ml}(2i\Gamma-\epsilon_{nl})^2}\end{aligned} \quad \text{(C-28)}$$

$$R_2 = -\frac{\beta^2\Gamma\psi^{(2)}(z_{m-})}{8\pi^3\epsilon_{ml}(2i\Gamma+\epsilon_{ml})}+\frac{\beta^2\Gamma\psi^{(2)}(z_{n+})}{8\pi^3\epsilon_{ln}(2i\Gamma-\epsilon_{ln})}+\frac{\beta^2\Gamma\psi^{(2)}(z_{l-})(-2i\Gamma+\epsilon_{nm})}{8\pi^3\epsilon_{nl}(-2i\Gamma+\epsilon_{nl})\epsilon_{lm}}+\frac{\beta^2\Gamma\psi^{(2)}(z_{l+})(2i\Gamma-\epsilon_{nm})}{8\pi^3\epsilon_{nl}\epsilon_{lm}(-2i\Gamma-\epsilon_{ml})} \quad \text{(C-29)}$$

The Taylor expansion of the coefficient with respect to $\Gamma$ is given by:

$$\begin{aligned}\mathcal{C}^{(2,2)}_{nml} =& \frac{i}{2}\Bigg(\psi^{(0)}(z^0_{n+})\frac{1}{\pi\epsilon_{nm}\epsilon^3_{nl}}-\psi^{(0)}(z^0_{n-})\frac{1}{\pi\epsilon_{nm}\epsilon^3_{nl}}+\psi^{(0)}(z^0_{m-})\frac{1}{\pi\epsilon_{nm}\epsilon^3_{ml}}-\psi^{(0)}(z^0_{m+})\frac{1}{\pi\epsilon_{nm}\epsilon^3_{ml}}\\ &+\psi^{(0)}(z^0_{l-})\frac{(\epsilon^2_{nm}+3\epsilon^2_{ml})}{\pi\epsilon^3_{nl}\epsilon^3_{lm}}-\psi^{(0)}(z^0_{l+})\frac{(\epsilon^2_{nm}+3\epsilon^2_{ml})}{\pi\epsilon^3_{nl}\epsilon^3_{lm}}\Bigg)+\mathcal{O}(\Gamma)\end{aligned} \quad \text{(C-30)}$$

Notably, the fully interband coefficient contributes exclusively to the intrinsic conductivity at $\mathcal{O}(\Gamma^0)$, representing a contribution that scales with the Fermi sea ($f_n$). This result is consistent with the multiband derivations provided in Supplementary Section E.

## Appendix D: Expansion of DC Conductivity in $\Gamma$

We now compute the DC conductivity $\sigma_{abc} \equiv \lim_{\omega\to 0} \sigma_{abc}(\omega, -\omega)$ for a generic two band model using expressions from Supplementary sections A, B, and C. We perform a Taylor expansion of the full DC conductivity $\sigma_{abc}$ in the relaxation rate $\Gamma$ around $\Gamma = 0$:

$$\sigma_{abc} = \frac{1}{\Gamma^2}\sigma^{(-2)}_{abc} + \frac{1}{\Gamma}\sigma^{(-1)}_{abc} + \sigma^{(0)}_{abc} + \mathcal{O}(\Gamma) \tag{D-1}$$

The coefficients $\sigma^{(k)}_{abc}$ are then independent of $\Gamma$. We analyze each of these coefficients for a two-band model ($n, m \in \{1, 2\}$). For conciseness we omit summation symbols for $\mathbf{k}$.

### 1. Notations and Key Identities for a Two-Band Model

To make the derivations self-contained, we first establish our notation and key identities, which strictly follow those in the provided analysis note.

*Definitions and Notation*

1. Band Energies: $\epsilon_n$. Energy difference: $\epsilon_{nm} = \epsilon_n - \epsilon_m$. We use $\bar{n}$ to denote the band other than $n$ (e.g., $\epsilon_{n\bar{n}} = \epsilon_n - \epsilon_{\bar{n}}$).
2. Velocity Matrix Elements: $v^a_{nm} = \langle u_n|\partial_{k_a} H|u_m\rangle$. The intraband velocity is $v^a_n \equiv v^a_{nn}$.
3. Interband Berry Connection: $A^a_{nm} = i\langle u_n|\partial_{k_a} u_m\rangle$ for $n \neq m$.
4. Distribution Function: Here we use the convention $f(\epsilon) = \tanh[(\beta/2)(\mu - \epsilon)]$, which comes from a complex conjugate pair of Polygamma functions in the $\Gamma \to 0$ limit. Its $k$-th derivative with respect to energy is $f^{(k)}_n \equiv d^k f_n/d\epsilon^k|_{\epsilon=\epsilon_n}$. This is related to the standard Fermi-Dirac distribution $f_0(\epsilon)$ by $f(\epsilon) = 2f_0(\epsilon) - 1$, and for derivatives $k \geq 1$, $f^{(k)}_n = 2f^{(k)}_{0,n}$. The final results in the main text are presented using $f_0$.
5. Quantum Metric: Here we use the convention $g_{ab} = \mathrm{Re}(A^a_{12}A^b_{21})$.

*Key Identities*

Our derivation relies on the following standard two-band model identities:

1. Feynman-Hellmann Identity: For $n \neq m$, $v^a_{nm} = -i\epsilon_{mn}A^a_{nm}$.
2. Metric-Velocity Relation: $v^a_{12}v^b_{21} + v^b_{12}v^a_{21} = \epsilon^2_{12}(A^a_{12}A^b_{21} + A^b_{12}A^a_{21}) = 2\epsilon^2_{12}g_{ab}$.
3. Diagonal Second Derivative: $v^{ab}_{nn} \equiv \langle u_n|\partial_{k_a}\partial_{k_b}H|u_n\rangle = \partial_a\partial_b\epsilon_n - 2\epsilon_{n\bar{n}}g_{ab}$.
4. Anti-symmetric Velocity Product: $V_{ab} \equiv v^a_{21}v^b_{12} - v^a_{12}v^b_{21} = i\epsilon^2_{12}\Omega^1_{ab}$, where $\Omega^1_{ab} = i(A^a_{12}A^b_{21} - A^b_{12}A^a_{21})$ is the Berry curvature of band 1.

### 2. The $\mathcal{O}(\Gamma^{-2})$ (Nonlinear Drude) Contribution

The $\sigma^{(-2)}_{abc}$ is proportional to $1/\Gamma^2$ and is

$$\sigma^{(-2)}_{abc} = \frac{1}{8\epsilon_{12}}[-K_1 f'_1 + K_2 f'_2] \tag{D-2}$$

The coefficients $K_1$ and $K_2$ are composed of two parts, involving interband velocities ($K_{nA}$) and intraband second derivatives ($K_{nB}$).

$$\begin{aligned} K_1 &= \underbrace{v_{21}^a(v_{12}^b v_1^c + v_1^b v_{12}^c) + v_{12}^a(v_{21}^b v_1^c + v_1^b v_{21}^c)}_{K_{1A}} + \underbrace{(v_{11}^{ac} v_1^b + v_{11}^{ab} v_1^c)\epsilon_{12}}_{K_{1B}} \\ K_2 &= \underbrace{v_{21}^a(v_2^b v_{12}^c + v_{12}^b v_2^c) + v_{12}^a(v_2^b v_{21}^c + v_{21}^b v_2^c)}_{K_{2A}} \underbrace{-(v_{22}^{ac} v_2^b + v_{22}^{ab} v_2^c)\epsilon_{12}}_{K_{2B}} \end{aligned} \tag{D-3}$$

We analyze the contributions from $K_A$ and $K_B$ separately.

*Analysis of $K_A$*

We reorganize $K_{1A}$ and apply the Metric-Velocity Relation ($v_{12}^a v_{21}^b + v_{12}^b v_{21}^a = 2\epsilon_{12}^2 g_{ab}$).

$$K_{1A} = v_1^c(v_{21}^a v_{12}^b + v_{12}^a v_{21}^b) + v_1^b(v_{21}^a v_{12}^c + v_{12}^a v_{21}^c) = 2\epsilon_{12}^2(v_1^c g_{ab} + v_1^b g_{ac}) \tag{D-4}$$

Similarly, $K_{2A} = 2\epsilon_{12}^2(v_2^c g_{ab} + v_2^b g_{ac})$. The contribution to $\sigma_{abc}^{(-2)}$ from these terms is $T_A^{(-2)}$:

$$T_A^{(-2)} = \frac{2\epsilon_{12}^2}{8\epsilon_{12}}[-(v_1^c g_{ab} + v_1^b g_{ac})f_1' + (v_2^c g_{ab} + v_2^b g_{ac})f_2'] = \frac{\epsilon_{12}}{4}[(g_{ab}v_2^c + g_{ac}v_2^b)f_2' - (g_{ab}v_1^c + g_{ac}v_1^b)f_1'] \tag{D-5}$$

*Analysis of $K_B$*

The contribution to $\sigma_{abc}^{(-2)}$ from the $K_B$ terms is $T_B^{(-2)}$:

$$\begin{aligned} T_B^{(-2)} &= \frac{1}{8\epsilon_{12}}[-K_{1B}f_1' + K_{2B}f_2'] = \frac{\epsilon_{12}}{8\epsilon_{12}}[-(v_{11}^{ac}v_1^b + v_{11}^{ab}v_1^c)f_1' - (v_{22}^{ac}v_2^b + v_{22}^{ab}v_2^c)f_2'] \\ &= -\frac{1}{8}\sum_n (v_{nn}^{ac}v_n^b + v_{nn}^{ab}v_n^c)f_n' \end{aligned} \tag{D-6}$$

We use the identity for the diagonal second derivative: $v_{nn}^{ab} = \partial_a\partial_b\epsilon_n - 2\epsilon_{n\bar{n}}g_{ab}$.

$$T_B^{(-2)} = -\frac{1}{8}\sum_n \left[(\partial_a\partial_c\epsilon_n - 2\epsilon_{n\bar{n}}g_{ac})v_n^b + (\partial_a\partial_b\epsilon_n - 2\epsilon_{n\bar{n}}g_{ab})v_n^c\right] f_n' \tag{D-7}$$

We separate this into an intraband part ($T_{B,\text{intra}}^{(-2)}$) and a geometric part ($T_{B,\text{geo}}^{(-2)}$):

$$T_B^{(-2)} = T_{B,\text{intra}}^{(-2)} + T_{B,\text{geo}}^{(-2)}, \tag{D-8}$$

with

$$T_{B,\text{intra}}^{(-2)} = -\frac{1}{8}\sum_n(\partial_a\partial_c\epsilon_n v_n^b + \partial_a\partial_b\epsilon_n v_n^c)f_n' = -\frac{1}{8}\sum_n \partial_a(v_n^b v_n^c)f_n' \tag{D-9}$$

and

$$T_{B,\text{geo}}^{(-2)} = -\frac{1}{8}\sum_n(-2\epsilon_{n\bar{n}})(g_{ac}v_n^b + g_{ab}v_n^c)f_n' = \frac{1}{4}\left[\epsilon_{12}(g_{ac}v_1^b + g_{ab}v_1^c)f_1' + \epsilon_{21}(g_{ac}v_2^b + g_{ab}v_2^c)f_2'\right] \tag{D-10}$$

Using $\epsilon_{21} = -\epsilon_{12}$:

$$T_{B,\text{geo}}^{(-2)} = \frac{\epsilon_{12}}{4}\left[(g_{ac}v_1^b + g_{ab}v_1^c)f_1' - (g_{ac}v_2^b + g_{ab}v_2^c)f_2'\right] \tag{D-11}$$

*Total Expression for $\sigma_{abc}^{(-2)}$*

We combine all contributions: $\sigma_{abc}^{(-2)} = T_A^{(-2)} + T_{B,\text{intra}}^{(-2)} + T_{B,\text{geo}}^{(-2)}$. Comparing the geometric terms $T_A^{(-2)}$ [Eq. (D-5) and $T_{B,\text{geo}}^{(-2)}$ [Eq. (D-11)], we find they are exactly opposite:

$$T_A^{(-2)} + T_{B,\text{geo}}^{(-2)} = 0 \tag{D-12}$$

The total conductivity $\sigma_{abc}^{(-2)}$ is therefore purely intraband:

$$\sigma_{abc}^{(-2)} = T_{B,\text{intra}}^{(-2)} = -\frac{1}{8}\sum_{n=1,2}\partial_a(v_n^b v_n^c)f_n', \qquad f_n' = 2f_{0,n}' \tag{D-13}$$

This result demonstrates that the leading divergence ($1/\Gamma^2$) depends only on the band structure derivatives (generalized Drude weight) and is independent of geometric quantities.

### 3. The $\mathcal{O}(\Gamma^{-1})$ (Berry Curvature Dipole) Contribution

The term proportional to $1/\Gamma$ is proportional to the first derivative of the distribution function, $f'$.

$$\sigma_{abc}^{(-1)} = -\frac{i}{4\epsilon_{12}^2}[C_1 f_1' + C_2 f_2'] \tag{D-14}$$

The coefficients $C_1$ and $C_2$ are given by:

$$\begin{aligned} C_1 &= v_{21}^a(v_{12}^b v_1^c + v_1^b v_{12}^c) - v_{12}^a(v_{21}^b v_1^c + v_1^b v_{21}^c) \\ C_2 &= -v_{21}^a(v_2^b v_{12}^c + v_{12}^b v_2^c) + v_{12}^a(v_2^b v_{21}^c + v_{21}^b v_2^c) \end{aligned} \tag{D-15}$$

We reorganize these expressions by factoring out the intraband velocities:

$$\begin{aligned} C_1 &= v_1^c(v_{21}^a v_{12}^b - v_{12}^a v_{21}^b) + v_1^b(v_{21}^a v_{12}^c - v_{12}^a v_{21}^c) \\ C_2 &= v_2^c(v_{12}^a v_{21}^b - v_{21}^a v_{12}^b) + v_2^b(v_{12}^a v_{21}^c - v_{21}^a v_{12}^c) \end{aligned} \tag{D-16}$$

The anti-symmetric structure suggests a connection to the Berry curvature. We analyze the velocity combination $V_{ab} \equiv v_{21}^a v_{12}^b - v_{12}^a v_{21}^b$. We utilize the Feynman-Hellmann identity ($v_{nm}^a = -i\epsilon_{mn}A_{nm}^a$).

$$\begin{aligned} V_{ab} &= (-i\epsilon_{12}A_{21}^a)(-i\epsilon_{21}A_{12}^b) - (-i\epsilon_{21}A_{12}^a)(-i\epsilon_{12}A_{21}^b) = (-1)(\epsilon_{12}\epsilon_{21})[A_{21}^a A_{12}^b - A_{12}^a A_{21}^b] \\ &= \epsilon_{12}^2(A_{21}^a A_{12}^b - A_{12}^a A_{21}^b) \qquad\qquad (\epsilon_{12}\epsilon_{21} = -\epsilon_{12}^2). \end{aligned} \tag{D-17}$$

The Berry curvature for band $n$ in a two-band system is defined as $\Omega_{ab}^n = i(A_{n\bar{n}}^a A_{\bar{n}n}^b - A_{n\bar{n}}^b A_{\bar{n}n}^a)$. For band 1: $\Omega_{ab}^1 = i(A_{12}^a A_{21}^b - A_{12}^b A_{21}^a)$. We relate the term in $V_{ab}$ to $\Omega_{ab}^1$:

$$A_{21}^a A_{12}^b - A_{12}^a A_{21}^b = -(A_{12}^a A_{21}^b - A_{12}^b A_{21}^a) = -(-i\Omega_{ab}^1) = i\Omega_{ab}^1 \tag{D-18}$$

Substituting this back into $V_{ab}$ [Eq. (D-17)]:

$$V_{ab} = i\epsilon_{12}^2\Omega_{ab}^1 \tag{D-19}$$

We substitute the expression for $V_{ab}$ back into the coefficients $C_1$ and $C_2$ [Eq. (D-16)].

$$C_1 = i\epsilon_{12}^2(v_1^c\Omega_{ab}^1 + v_1^b\Omega_{ac}^1), \qquad C_2 = -i\epsilon_{12}^2(v_2^c\Omega_{ab}^1 + v_2^b\Omega_{ac}^1) \tag{D-20}$$

Finally, we substitute $C_1$ and $C_2$ into the expression for $\sigma_{abc}^{(-1)}$ [Eq. (D-14)]:

$$\begin{aligned} \sigma_{abc}^{(-1)} &= -\frac{i}{4\epsilon_{12}^2}(i\epsilon_{12}^2)\left[(v_1^c\Omega_{ab}^1 + v_1^b\Omega_{ac}^1)f_1' - (v_2^c\Omega_{ab}^1 + v_2^b\Omega_{ac}^1)f_2'\right] \\ &= \frac{1}{4}\left[(v_1^c\Omega_{ab}^1 + v_1^b\Omega_{ac}^1)f_1' - (v_2^c\Omega_{ab}^1 + v_2^b\Omega_{ac}^1)f_2'\right] \end{aligned} \tag{D-21}$$

Using the property that the Berry curvature is opposite for the two bands ($\Omega_{ab}^2 = -\Omega_{ab}^1$), we can write this compactly:

$$\sigma_{abc}^{(-1)} = \frac{1}{4}\sum_{n=1,2}(v_n^c\Omega_{ab}^n + v_n^b\Omega_{ac}^n)f_n', \qquad f_n' = 2f_{0,n}' \tag{D-22}$$

This term is a Fermi surface contribution related to the nonlinear anomalous Hall effect (NAHE).

### 4. The $\mathcal{O}(\Gamma^0)$ (Intrinsic) Contribution

This is the central result. The $\Gamma^0$ term, $\sigma^{(0)}_{abc}$, is composed of five distinct terms from the $\Gamma$-expansion, which we group based on the power of the inverse temperature $\beta$: $\sigma^{(0)}_{abc} = T_0 + T_1 + T_2 + T_3 + T_4$.

*Term 4 ($\sim \beta^4$, $f^{(4)}$)*

This term is proportional to the fourth derivative of $f(\epsilon)$:

$$T_4 = \sum_{n=1,2} \frac{1}{4!} v_n^a v_n^b v_n^c f_n^{(4)} = \sum_{n=1,2} \frac{1}{24} v_n^a v_n^b v_n^c f_n^{(4)}, \tag{D-23}$$

which is a purely intraband contribution.

*Term 3 ($\sim \beta^3$, $f^{(3)}$)*

This term is proportional to the third derivative.

$$T_3 = \frac{1}{48\epsilon_{12}} \left[ K_1 f_1^{(3)} - K_2 f_2^{(3)} \right] \tag{D-24}$$

The coefficients $K_n$ contain terms involving interband velocities ($K_{nA}$) and second derivative matrix elements ($K_{nB}$).

$$\begin{aligned} K_1 &= \underbrace{v_{21}^a (v_{12}^b v_1^c + v_1^b v_{12}^c) + v_{12}^a (v_{21}^b v_1^c + v_1^b v_{21}^c)}_{K_{1A}} + \underbrace{(v_{11}^{ac} v_1^b + v_{11}^{ab} v_1^c)\epsilon_{12}}_{K_{1B}} \\ K_2 &= \underbrace{v_{21}^a (v_2^b v_{12}^c + v_{12}^b v_2^c) + v_{12}^a (v_2^b v_{21}^c + v_{21}^b v_2^c)}_{K_{2A}} \underbrace{-(v_{22}^{ac} v_2^b + v_{22}^{ab} v_2^c)\epsilon_{12}}_{K_{2B}} \end{aligned} \tag{D-25}$$

We analyze the contributions $T_{3A}$ and $T_{3B}$ separately. Using the Metric-Velocity Relation, $K_{1A} = 2\epsilon_{12}^2 (v_1^c g_{ab} + v_1^b g_{ac})$ and $K_{2A} = 2\epsilon_{12}^2 (v_2^c g_{ab} + v_2^b g_{ac})$.

For $T_{3A}$,

$$T_{3A} = \frac{2\epsilon_{12}^2}{48\epsilon_{12}} \left[ (v_1^c g_{ab} + v_1^b g_{ac}) f_1^{(3)} - (v_2^c g_{ab} + v_2^b g_{ac}) f_2^{(3)} \right] = \frac{\epsilon_{12}}{24} \left[ (g_{ab} v_1^c + g_{ac} v_1^b) f_1^{(3)} - (g_{ab} v_2^c + g_{ac} v_2^b) f_2^{(3)} \right] \tag{D-26}$$

For $T_{3B}$,

$$\begin{aligned} T_{3B} &= \frac{1}{48\epsilon_{12}} \left[ K_{1B} f_1^{(3)} - K_{2B} f_2^{(3)} \right] = \frac{\epsilon_{12}}{48\epsilon_{12}} \left[ (v_{11}^{ac} v_1^b + v_{11}^{ab} v_1^c) f_1^{(3)} + (v_{22}^{ac} v_2^b + v_{22}^{ab} v_2^c) f_2^{(3)} \right] \\ &= \frac{1}{48} \sum_n (v_{nn}^{ac} v_n^b + v_{nn}^{ab} v_n^c) f_n^{(3)} \end{aligned} \tag{D-27}$$

We utilize the Diagonal Second Derivative Identity: $v_{nn}^{ab} = \partial_a \partial_b \epsilon_n - 2\epsilon_{n\bar{n}} g_{ab}$.

$$T_{3B} = \frac{1}{48} \sum_n \left[ (\partial_a \partial_c \epsilon_n - 2\epsilon_{n\bar{n}} g_{ac}) v_n^b + (\partial_a \partial_b \epsilon_n - 2\epsilon_{n\bar{n}} g_{ab}) v_n^c \right] f_n^{(3)}, \tag{D-28}$$

and separate this into intraband derivative terms ($T_{3B}^{\text{intra}}$) and geometric terms ($T_{3B}^{\text{inter}}$):

$$T_{3B}^{\text{intra}} = \frac{1}{48} \sum_n (\partial_a \partial_c \epsilon_n v_n^b + \partial_a \partial_b \epsilon_n v_n^c) f_n^{(3)} \tag{D-29}$$

$$T_{3B}^{\text{inter}} = -\frac{2}{48} \sum_n \epsilon_{n\bar{n}} (g_{ac} v_n^b + g_{ab} v_n^c) f_n^{(3)} \tag{D-30}$$

Since $\epsilon_{21} = -\epsilon_{12}$, we find $T_{3B}^{\text{inter}} = -T_{3A}$.

For Total $T_3$, the geometric contributions cancel exactly: $T_3 = T_{3A} + T_{3B} = T_{3B}^{\text{intra}}$. We recognize the intraband term as a total derivative: $\partial_a \partial_c \epsilon_n v_n^b + \partial_a \partial_b \epsilon_n v_n^c = \partial_a (v_n^b v_n^c)$.

$$T_3 = \frac{1}{48} \sum_{n=1,2} \partial_a (v_n^b v_n^c) f_n^{(3)} \tag{D-31}$$

This is also a purely intraband contribution.

*Term 2 ($\sim \beta^2$, $f^{(2)}$)*

This term is proportional to the second derivative.

$$T_2 = \frac{1}{4\epsilon_{12}^2} \sum_n v_n^a (v_{21}^b v_{12}^c + v_{12}^b v_{21}^c) f_n^{(2)} \tag{D-32}$$

Applying the Metric-Velocity Relation:

$$T_2 = \frac{1}{4\epsilon_{12}^2} \sum_n v_n^a (2\epsilon_{12}^2 g_{bc}) f_n^{(2)} = \frac{1}{2} g_{bc} \sum_n v_n^a f_n^{(2)} \tag{D-33}$$

*Term 1 ($\sim \beta^1$, $f^{(1)}$)*

This term is proportional to the first derivative. By matching coefficients:

$$T_1 = \frac{1}{2\epsilon_{12}^3} \left[ C_1 f_1^{(1)} - C_2 f_2^{(1)} \right] \tag{D-34}$$

The coefficients $C_1$ and $C_2$ are:

$$\begin{aligned} C_1 &= v_{21}^a (v_{12}^b v_1^c + v_1^b v_{12}^c) + v_{12}^a (v_{21}^b v_1^c + v_1^b v_{21}^c) - v_1^a (v_{21}^b v_{12}^c + v_{12}^b v_{21}^c) \\ C_2 &= v_{21}^a (v_2^b v_{12}^c + v_{12}^b v_2^c) + v_{12}^a (v_2^b v_{21}^c + v_{21}^b v_2^c) - v_2^a (v_{21}^b v_{12}^c + v_{12}^b v_{21}^c) \end{aligned} \tag{D-35}$$

We reorganize $C_1$ and apply the Metric-Velocity Relation:

$$\begin{aligned} C_1 &= v_1^c (v_{21}^a v_{12}^b + v_{12}^a v_{21}^b) + v_1^b (v_{21}^a v_{12}^c + v_{12}^a v_{21}^c) - v_1^a (v_{21}^b v_{12}^c + v_{12}^b v_{21}^c) \\ &= v_1^c (2\epsilon_{12}^2 g_{ab}) + v_1^b (2\epsilon_{12}^2 g_{ac}) - v_1^a (2\epsilon_{12}^2 g_{bc}) \end{aligned} \tag{D-36}$$

$C_2$ follows similarly. Substituting these back into $T_1$:

$$\begin{aligned} T_1 &= \frac{2\epsilon_{12}^2}{2\epsilon_{12}^3} \left[ (v_1^c g_{ab} + v_1^b g_{ac} - v_1^a g_{bc}) f_1' - (v_2^c g_{ab} + v_2^b g_{ac} - v_2^a g_{bc}) f_2' \right] \\ &= \frac{1}{\epsilon_{12}} \left[ (v_1^c g_{ab} + v_1^b g_{ac} - v_1^a g_{bc}) f_1' - (v_2^c g_{ab} + v_2^b g_{ac} - v_2^a g_{bc}) f_2' \right] \end{aligned} \tag{D-37}$$

*Term 0 ($\sim \beta^0$, $f$)*

This term represents a Fermi sea contribution at first glance:

$$T_0 = \frac{f_1 - f_2}{2\epsilon_{12}^4} \times C_0 \tag{D-38}$$

The coefficient $C_0$ is decomposed into three parts ($A$, $B$, $E$):

$$A = 3(v_2^a - v_1^a)(v_{21}^b v_{12}^c + v_{12}^b v_{21}^c)$$
$$B = (v_2^b - v_1^b)(v_{21}^a v_{12}^c + v_{12}^a v_{21}^c) + (v_2^c - v_1^c)(v_{21}^a v_{12}^b + v_{12}^a v_{21}^b) \tag{D-39}$$
$$E = (v_{21}^{ac} v_{12}^b + v_{12}^{ac} v_{21}^b + v_{21}^{ab} v_{12}^c + v_{12}^{ab} v_{21}^c)\epsilon_{12}$$

For Part A and Part B we use Metric-Velocity Relation

$$A = 3(v_2^a - v_1^a)(2\epsilon_{12}^2 g_{bc}) = 6\epsilon_{12}^2 g_{bc}(v_2^a - v_1^a) \tag{D-40}$$

$$B = 2\epsilon_{12}^2[g_{ac}(v_2^b - v_1^b) + g_{ab}(v_2^c - v_1^c)] \tag{D-41}$$

For Part E, we define $S_{ab;c} = v_{21}^{ab} v_{12}^c + v_{12}^{ab} v_{21}^c$. Then $E = (S_{ac;b} + S_{ab;c})\epsilon_{12}$. We use the Off-Diagonal Second Derivative Identity: $v_{nm}^{ab} = \partial_a v_{nm}^b - i[A^a, v^b]_{nm}$.

$$S_{ab;c} = (\partial_a v_{21}^b) v_{12}^c + (\partial_a v_{12}^b) v_{21}^c - iK_{ab;c} = \partial_a (v_{21}^b v_{12}^c) - iK_{ab;c} \tag{D-42}$$

where $K_{ab;c} = [A^a, v^b]_{21} v_{12}^c + [A^a, v^b]_{12} v_{21}^c$. We consider the symmetric combination required for E:

$$S_{ac;b} + S_{ab;c} = \partial_a (v_{21}^c v_{12}^b + v_{21}^b v_{12}^c) - iK_{abc} \tag{D-43}$$

where $K_{abc} = K_{ac;b} + K_{ab;c}$. We use Metric-Velocity Relation for the term inside the derivative:

$$S_{ac;b} + S_{ab;c} = \partial_a (2\epsilon_{12}^2 g_{bc}) - iK_{abc} \tag{D-44}$$

We evaluate the commutator term $K_{abc}$ and check its gauge invariance. In a two-band system, the commutators are:

$$[A^a, v^b]_{12} = A_{12}^a (v_2^b - v_1^b) + v_{12}^b (A_{11}^a - A_{22}^a)$$
$$[A^a, v^b]_{21} = A_{21}^a (v_1^b - v_2^b) + v_{21}^b (A_{22}^a - A_{11}^a) \tag{D-45}$$

Substituting these into the definition of $K_{ab;c}$:

$$K_{ab;c} = (v_2^b - v_1^b)(A_{12}^a v_{21}^c - A_{21}^a v_{12}^c) + (A_{11}^a - A_{22}^a)(v_{12}^b v_{21}^c - v_{21}^b v_{12}^c) \tag{D-46}$$

When calculating $K_{abc} = K_{ac;b} + K_{ab;c}$, the gauge-dependent part (proportional to $A_{11}^a - A_{22}^a$) is:

$$P_{\text{gauge}} = (A_{11}^a - A_{22}^a)[(v_{12}^c v_{21}^b - v_{21}^c v_{12}^b) + (v_{12}^b v_{21}^c - v_{21}^b v_{12}^c)] \tag{D-47}$$

Since the velocity matrix elements commute ($v_{nm}^a v_{kl}^b = v_{kl}^b v_{nm}^a$), the terms inside the bracket cancel exactly: $P_{\text{gauge}} = 0$. This explicitly confirms the gauge invariance of E.

The remaining gauge-invariant part of $K_{abc}$ is:

$$K_{abc} = (v_2^c - v_1^c)(A_{12}^a v_{21}^b - A_{21}^a v_{12}^b) + (v_2^b - v_1^b)(A_{12}^a v_{21}^c - A_{21}^a v_{12}^c) \tag{D-48}$$

We evaluate the bracketed terms using the Feynman-Hellmann identity.

$$A_{12}^a v_{21}^b - A_{21}^a v_{12}^b = A_{12}^a(-i\epsilon_{12} A_{21}^b) - A_{21}^a(-i\epsilon_{21} A_{12}^b) = -i\epsilon_{12}(A_{12}^a A_{21}^b + A_{21}^a A_{12}^b) = -2i\epsilon_{12} g_{ab} \tag{D-49}$$

Substituting this back into $K_{abc}$:

$$K_{abc} = -2i\epsilon_{12}\left[(v_2^c - v_1^c) g_{ab} + (v_2^b - v_1^b) g_{ac}\right] \tag{D-50}$$

Now we assemble the expression for $S_{ac;b} + S_{ab;c}$ (Eq. 28):

$$S_{ac;b} + S_{ab;c} = \partial_a (2\epsilon_{12}^2 g_{bc}) - iK_{abc} = \partial_a (2\epsilon_{12}^2 g_{bc}) - 2\epsilon_{12}\left[g_{ab}(v_2^c - v_1^c) + g_{ac}(v_2^b - v_1^b)\right] \tag{D-51}$$

We expand the derivative term, using $\partial_a \epsilon_{12} = v_1^a - v_2^a$:

$$\partial_a (2\epsilon_{12}^2 g_{bc}) = 4\epsilon_{12}(\partial_a \epsilon_{12}) g_{bc} + 2\epsilon_{12}^2(\partial_a g_{bc}) = 4\epsilon_{12}(v_1^a - v_2^a) g_{bc} + 2\epsilon_{12}^2(\partial_a g_{bc}) \tag{D-52}$$

The complete expression for $E = (S_{ac;b} + S_{ab;c})\epsilon_{12}$ is:

$$E = 4\epsilon_{12}^2(v_1^a - v_2^a)g_{bc} + 2\epsilon_{12}^3(\partial_a g_{bc}) - 2\epsilon_{12}^2\left[g_{ab}(v_2^c - v_1^c) + g_{ac}(v_2^b - v_1^b)\right] \tag{D-53}$$

We sum the results A [Eq. (D-40)], B [Eq. (D-41)], and E [Eq. (D-53)]:

$$C_0 = \underbrace{6\epsilon_{12}^2 g_{bc}(v_2^a - v_1^a)}_{A} + \underbrace{2\epsilon_{12}^2[g_{ac}(v_2^b - v_1^b) + g_{ab}(v_2^c - v_1^c)]}_{B} + E \tag{D-54}$$

We group the terms by the geometric quantities:

$$\begin{aligned} C_0 = {} & \epsilon_{12}^2 g_{bc}[6(v_2^a - v_1^a) + 4(v_1^a - v_2^a)] \\ & + \epsilon_{12}^2 g_{ac}[2(v_2^b - v_1^b) - 2(v_2^b - v_1^b)] \\ & + \epsilon_{12}^2 g_{ab}[2(v_2^c - v_1^c) - 2(v_2^c - v_1^c)] + 2\epsilon_{12}^3(\partial_a g_{bc}) \end{aligned} \tag{D-55}$$

The terms proportional to $g_{ac}$ and $g_{ab}$ cancel exactly.

$$C_0 = 2\epsilon_{12}^2 g_{bc}(v_2^a - v_1^a) + 2\epsilon_{12}^3(\partial_a g_{bc}) \tag{D-56}$$

Substituting the simplified $C_0$ back into $T_0$ [Eq. (D-38)]:

$$T_0 = (f_1 - f_2)\left[\frac{g_{bc}(v_2^a - v_1^a)}{\epsilon_{12}^2} + \frac{\partial_a g_{bc}}{\epsilon_{12}}\right] \tag{D-57}$$

Since $v_2^a - v_1^a = -(\partial_a \epsilon_{12})$, this expression is recognized as the quotient rule for a total derivative:

$$T_0 = (f_1 - f_2)\left[\frac{(\partial_a g_{bc})\epsilon_{12} - g_{bc}(\partial_a \epsilon_{12})}{\epsilon_{12}^2}\right] = (f_1 - f_2)\partial_a\left(\frac{g_{bc}}{\epsilon_{12}}\right). \tag{D-58}$$

## III. Transformation and Final Expressions

We use integration by parts (IBP) in $k$-space, $\int(\partial_a A)B = -\int A(\partial_a B)$, to simplify the expressions and group them into physical Fermi surface contributions.

### *A. Kinetic Contribution*

The total kinetic contribution is $\sigma_{abc}^{\text{kin}} = T_3 + T_4$. We apply IBP to $T_3$ [Eq. (D-31)] with respect to $k_a$.

$$T_3 \quad \xrightarrow{\text{IBP}} \quad -\frac{1}{48}\sum_n (v_n^b v_n^c)\partial_a(f_n^{(3)}) \tag{D-59}$$

Using the chain rule $\partial_a f_n^{(3)} = f_n^{(4)} v_n^a$:

$$T_3 \quad \xrightarrow{\text{IBP}} \quad -\frac{1}{48}\sum_n v_n^a v_n^b v_n^c f_n^{(4)} = -\frac{1}{2}T_4 \tag{D-60}$$

The total intraband contribution simplifies to:

$$\sigma_{abc}^{\text{kin}} = T_3 + T_4 = \frac{1}{2}T_4 = \sum_{n=1,2}\frac{1}{48}v_n^a v_n^b v_n^c f_n^{(4)} = \sum_{n=1,2}\frac{1}{24}v_n^a v_n^b v_n^c f_{0,n}^{(4)}, \quad f_n' = 2f_{0,n}' \tag{D-61}$$

This result highlights that the specific mechanism of dissipation (fermionic bath) dictates the NESS structure such that a purely kinetic, ballistic response survives in the intrinsic $\left(\Gamma^0\right)$ limit, originating from the Polygamma functions discussed in Section C 2 a. This contrasts with models assuming RTAs.

To improve numerical stability, we can reduce the order of the derivative on the Fermi function by integration by parts over the Brillouin Zone, and rewrite $\sigma_{abc}^{\text{kin}}$ as:

$$\sigma_{abc}^{\text{kin}} = \sum_n \frac{1}{12}\left(\frac{\partial^3 \epsilon_n}{\partial k_a \partial k_b \partial k_c}\right) f_{0,n}^{(2)}. \tag{D-62}$$

### *B. Geometric Contribution*

We apply IBP to $T_0$ [Eq. (D-58)] to transform the Fermi sea term to the Fermi surface:

$$T_0 \quad \xrightarrow{\text{IBP}} \quad -\left[\partial_a(f_1 - f_2)\right]\left(\frac{g_{bc}}{\epsilon_{12}}\right) = -(f_1' v_1^a - f_2' v_2^a)\left(\frac{g_{bc}}{\epsilon_{12}}\right) = g_{bc}\left(\frac{v_2^a f_2' - v_1^a f_1'}{\epsilon_{12}}\right), \tag{D-63}$$

The total interband Fermi surface contribution is $\sigma_{abc}^{\text{inter-FS}} = T_0 + T_1 + T_2$. We combine the $f'$ terms from $T_0$ and $T_1$ [Eq. (D-37)]:

$$T_0 + T_1 = \frac{1}{\epsilon_{12}}\left[g_{bc}(v_2^a f_2' - v_1^a f_1') + (v_1^c g_{ab} + v_1^b g_{ac} - v_1^a g_{bc})f_1' - (v_2^c g_{ab} + v_2^b g_{ac} - v_2^a g_{bc})f_2'\right] \tag{D-64}$$

Grouping terms by $f_1'$ and $f_2'$:

$$T_0 + T_1 = \frac{1}{\epsilon_{12}}\left[f_1'(v_1^c g_{ab} + v_1^b g_{ac} - 2v_1^a g_{bc}) - f_2'(v_2^c g_{ab} + v_2^b g_{ac} - 2v_2^a g_{bc})\right] \tag{D-65}$$

The total geometric contribution, including $T_2$ [Eq. (D-33)], is:

$$\sigma_{abc}^{\text{geo}} = \frac{1}{2} g_{bc} \sum_n v_n^a f_n'' + (T_0 + T_1) \tag{D-66}$$

We transform the $f''$ term ($T_2$) into an $f'$ term using IBP. We use the identity $v_n^a f_n^{(2)} = \partial_a(f_n')$.

$$T_2 = \sum_n (\frac{1}{2} g_{bc})\partial_a(f_n') \tag{D-67}$$

Applying IBP w.r.t $k_a$:

$$T_2 \quad \xrightarrow{\text{IBP}} \quad -\sum_n \left[\partial_a(\frac{1}{2}g_{bc})\right] f_n' = -\frac{1}{2}(\partial_a g_{bc})(f_1' + f_2') \tag{D-68}$$

This transformation explicitly reveals the intraband quantum metric dipole contribution, $-\partial_a g_{bc}/2$. Physically, this term corresponds to the Fermi surface contribution arising from the gradient of the quantum geometry, which governs the deformation and geometric shift of wave packets under the external field.

Substituting this transformed $T_2$ and $(T_0 + T_1)$ [Eq. (D-64)] into $\sigma_{abc}^{\text{geo}}$ and grouping terms by $f_n'$:

$$\begin{aligned} \sigma_{abc}^{\text{geo}} = \quad & f_1'\left[-\frac{1}{2}(\partial_a g_{bc}) + \frac{1}{\epsilon_{12}}(v_1^c g_{ab} + v_1^b g_{ac} - 2v_1^a g_{bc})\right] \\ & + f_2'\left[-\frac{1}{2}(\partial_a g_{bc}) - \frac{1}{\epsilon_{12}}(v_2^c g_{ab} + v_2^b g_{ac} - 2v_2^a g_{bc})\right] \end{aligned} \tag{D-69}$$

We rewrite the expression for band 2 using the energy denominator $\epsilon_{21} = -\epsilon_{12}$ to achieve a symmetric form:

$$\begin{aligned} \sigma_{abc}^{\text{geo}} = \quad & f_1'\left[-\frac{1}{2}(\partial_a g_{bc}) + \frac{1}{\epsilon_{12}}(g_{ab} v_1^c + g_{ac} v_1^b - 2g_{bc} v_1^a)\right] \\ & + f_2'\left[-\frac{1}{2}(\partial_a g_{bc}) + \frac{1}{\epsilon_{21}}(g_{ab} v_2^c + g_{ac} v_2^b - 2g_{bc} v_2^a)\right] \end{aligned} \tag{D-70}$$

This can be written compactly using $\epsilon_{n\bar{n}}$ (where $\epsilon_{1\bar{1}} = \epsilon_{12}$ and $\epsilon_{2\bar{2}} = \epsilon_{21}$):

$$\sigma_{abc}^{\text{geo}} = \sum_{n=1,2} 2f_{0,n}' \left[-\frac{1}{2}(\partial_a g_{bc}) + \frac{1}{\epsilon_{n\bar{n}}}(g_{ab} v_n^c + g_{ac} v_n^b - 2g_{bc} v_n^a)\right], \quad f_n' = 2f_{0,n}' \tag{D-71}$$

## Appendix E: DC Conductivity of Multiband Model

We now consider a general multiband model to calculate the DC conductivity $\sigma_{abc} = \lim_{\omega\to 0}\sigma_{abc}(\omega,-\omega)$, and demonstrate that the results obtained for the two-band model in Supplementary Section D can be generalized to an multiband system. The derivation procedure remains essentially similar to that presented in Supplementary Section D. In the following derivation, we utilize abstract band indices ($n, l, m \in \{1, 2, ..., N\}$), rather than the specific indices.

### 1. Notation and Key Identities for a Multiband Model

In Supplementary Section D 1, we defined the notations and key identities specific to the two-band model. However, to investigate multiband systems, this formalism must be further generalized. We summarize the necessary modifications and extensions below:

*Definitions and Notation*

1. Group Velocity Difference: $\Delta^a_{nm} = v^a_n - v^a_m$.
2. Summation over band indices: $\sum'_I$, where $I$ represents the set of indices to be summed, and the prime ($'$) in the superscript denotes that all indices in the expression must be mutually distinct.
3. Multiband Berry Curvature: $\Omega^{ab}_n = i\sum'_m (A^a_{nm}A^b_{mn} - A^b_{nm}A^a_{mn})$.
4. Multiband Quantum Metric: $g^{ab}_n = \sum'_m \mathrm{Re}(A^a_{nm}A^b_{mn})$.
5. Multiband Band-renormalized Quantum Metric: $\mathcal{G}^{ab}_n = \sum'_m \mathrm{Re}(A^a_{nm}A^b_{mn}/\epsilon_{nm})$.

*Key Identities*

1. Multiband Metric-Velocity Relation: $\sum'_m (v^a_{nm}v^b_{mn} + v^b_{nm}v^a_{mn})/\epsilon^2_{nm} = 2g^{ab}_n$.
2. Multiband Band-renormalized Metric-Velocity Relation: $\sum'_m (v^a_{nm}v^b_{mn} + v^b_{nm}v^a_{mn})/\epsilon^3_{nm} = 2\mathcal{G}^{ab}_n$.
3. Multiband Diagonal Second Derivative:
$$v^{ab}_{nn} = \partial_a\partial_b\epsilon_n - \sum'_m \frac{v^a_{nm}v^b_{mn} + v^b_{nm}v^a_{mn}}{\epsilon_{nm}}. \tag{E-1}$$
4. Multiband Off-diagonal Second Derivative:
$$v^{ab}_{nm} = -iR^{ab}_{nm}v^b_{nm} + \frac{v^b_{nm}\Delta^a_{nm}}{\epsilon_{nm}} + \frac{v^a_{nm}\Delta^b_{nm}}{\epsilon_{nm}} - \sum'_l\Big(\frac{v^a_{nl}v^b_{lm}}{\epsilon_{nl}} - \frac{v^b_{nl}v^a_{lm}}{\epsilon_{lm}}\Big). \tag{E-2}$$
where the shift vector is defined as $R^{ab}_{nm} = i\partial_a \ln A^b_{nm} + A^a_{nn} - A^a_{mm}$.
5. Multiband Curvature-Velocity Relation: $\sum'_m (v^a_{nm}v^b_{mn} - v^b_{nm}v^a_{mn})/\epsilon^2_{nm} = -i\Omega^{ab}_n$.

### 2. The $\mathcal{O}(\Gamma^{-2})$ (Nonlinear Drude) Contribution

The $\Gamma^{-2}$ contribution, $\sigma^{(-2)}_{abc}$ is given by:

$$\sigma^{(-2)}_{abc} = -\frac{1}{8}\sum_n f^{(1)}_n\Big[v^{ac}_{nn}v^b_n + v^{ab}_{nn}v^c_n + \sum'_m\Big(v^c_n\frac{v^a_{mn}v^b_{nm} + v^b_{mn}v^a_{nm}}{\epsilon_{nm}} + v^b_n\frac{v^a_{mn}v^c_{nm} + v^c_{mn}v^a_{nm}}{\epsilon_{nm}}\Big)\Big] \tag{E-3}$$

By using the property of Diagonal Second Derivative, the expression can be significantly simplified to:

$$\sigma^{(-2)}_{abc} = -\frac{1}{8}\sum_n f^{(1)}_n\Big(\partial_a\partial_c\epsilon_n v^b_n + \partial_a\partial_b\epsilon_n v^c_n\Big) = -\frac{1}{8}\sum_n \partial_a(v^b_n v^c_n)f^{(1)}_n \tag{E-4}$$

Since the Drude contribution involves only intrinsic properties, this result constitutes a direct multiband extension of the two-band model.

### 3. The $\mathcal{O}(\Gamma^{-1})$ (Berry Curvature Dipole) Contribution

The $\Gamma^{-1}$ contribution, $\sigma^{(-1)}_{abc}$, is derived as:

$$\sigma^{(-1)}_{abc} = -\frac{i}{4}\sum\nolimits'_{nm} f^{(1)}_n \left( v^c_n \frac{v^a_{mn} v^b_{nm} - v^b_{mn} v^a_{nm}}{\epsilon^2_{nm}} + v^b_n \frac{v^a_{mn} v^c_{nm} - v^c_{mn} v^a_{nm}}{\epsilon^2_{nm}} \right) \tag{E-5}$$

By utilizing the multiband curvature-velocity relation, we can express this contribution in terms of the Berry curvature:

$$\sigma^{(-1)}_{abc} = \frac{1}{4}\sum\nolimits_n f^{(1)}_n \left( v^c_n \Omega^{ab}_n + v^b_n \Omega^{ac}_n \right) \tag{E-6}$$

This result represents a natural generalization of the two-band case, achieved by replacing the two-band Berry curvature (defined via the anti-symmetric velocity product) with its multiband counterpart.

### 4. The $\mathcal{O}(\Gamma^0)$ (Intrinsic) Contribution

Following the calculation scheme in Supplementary Section D 4, we expand $\sigma^{(0)}_{abc}$ in powers of the inverse temperature $\beta$:

$$\sigma^{(0)}_{abc} = T_0 + T_1 + T_2 + T_3 + T_4, \tag{E-7}$$

which we investigate term by term below.

*Term 4 ($\sim \beta^4$, $f^{(4)}$)*

This contribution is proportional to the fourth derivative of Fermi-Dirac distribution:

$$T_4 = \frac{1}{4!}\sum\nolimits_n v^a_n v^b_n v^c_n f^{(4)}_n \tag{E-8}$$

*Term 3 ($\sim \beta^3$, $f^{(3)}$)*

This contribution is proportional to the third derivative of Fermi-Dirac distribution:

$$T_3 = \sum\nolimits_n \frac{f^{(3)}_n}{2\times 4!}\left[ v^{ac}_{nn} v^b_n + v^{ab}_{nn} v^c_n + \sum\nolimits'_m \left( v^b_n \frac{v^a_{nm} v^c_{mn} + v^c_{nm} v^a_{mn}}{\epsilon_{nm}} + v^c_n \frac{v^a_{nm} v^b_{mn} + v^b_{nm} v^a_{mn}}{\epsilon_{nm}} \right)\right] \tag{E-9}$$

Then, we utilize the property of Diagonal Second Derivative to simplify the expression:

$$T_3 = \sum\nolimits_n \frac{1}{2\times 4!}\left( \partial_a \partial_c \epsilon_n v^b_n + \partial_a \partial_b \epsilon_n v^c_n \right) f^{(3)}_n = \sum\nolimits_n \frac{1}{2\times 4!}\partial_a (v^c_n v^b_n) f^{(3)}_n \tag{E-10}$$

*Term 2 ($\sim \beta^2$, $f^{(2)}$)*

This contribution is proportional to the second derivative of Fermi-Dirac distribution:

$$T_2 = \frac{1}{4}\sum\nolimits'_{nm} f^{(2)}_n v^a_n \frac{v^b_{nm} v^c_{mn} + v^c_{nm} v^b_{mn}}{\epsilon^2_{nm}} \tag{E-11}$$

By using the Multiband Metric-Velocity Relation:

$$T_2 = \frac{1}{2}\sum\nolimits_n f^{(2)}_n v^a_n g^{bc}_n \tag{E-12}$$

*Term 1 ($\sim \beta^1$, $f^{(1)}$)*

This contribution is proportional to the first derivative of Fermi-Dirac distribution:

$$T_1 = \sum_{nm}' \frac{1}{2} f_n^{(1)} \Big[ v_n^b \frac{v_{nm}^a v_{mn}^c + v_{nm}^c v_{mn}^a}{\epsilon_{nm}^3} + v_n^c \frac{v_{nm}^a v_{mn}^b + v_{nm}^b v_{mn}^a}{\epsilon_{nm}^3} - v_n^a \frac{v_{nm}^b v_{mn}^c + v_{nm}^c v_{mn}^b}{\epsilon_{nm}^3} \Big] \tag{E-13}$$

Applying the Multiband Band-renormalized Metric-Velocity Relation:

$$T_1 = \sum_n f_n^{(1)} \Big[ v_n^b \mathcal{G}_n^{ac} + v_n^c \mathcal{G}_n^{ab} - v_n^a \mathcal{G}_n^{bc} \Big] \tag{E-14}$$

*Term 0 ($\sim \beta^0$, $f^{(0)}$)*

Up to this point, our calculations for the multiband model show that terms $T_1$ through $T_4$ are consistent with the two-band model. While $T_0$ appears to introduce additional multiband contributions, we can demonstrate that these extra terms vanish. The explicit form of $T_0$ is:

$$\begin{aligned} T_0 = \sum_{nm}' \frac{1}{2} f_n \Big[ & 3\Delta_{mn}^a \frac{v_{nm}^b v_{mn}^c + v_{nm}^c v_{mn}^b}{\epsilon_{nm}^4} + \Delta_{mn}^b \frac{v_{nm}^a v_{mn}^c + v_{nm}^c v_{mn}^a}{\epsilon_{nm}^4} + \Delta_{mn}^c \frac{v_{nm}^a v_{mn}^b + v_{nm}^b v_{mn}^a}{\epsilon_{nm}^4} \\ & + \sum_l' \Big( \frac{v_{ln}^a v_{nm}^b v_{ml}^c}{\epsilon_{ln}\epsilon_{mn}^3} + \frac{v_{ln}^a v_{nm}^c v_{ml}^b}{\epsilon_{ln}\epsilon_{mn}^3} + \frac{v_{nl}^a v_{lm}^c v_{mn}^b}{\epsilon_{ln}\epsilon_{mn}^3} + \frac{v_{nl}^a v_{lm}^b v_{mn}^c}{\epsilon_{ln}\epsilon_{mn}^3} - \frac{v_{ml}^a v_{ln}^b v_{nm}^c}{\epsilon_{ml}\epsilon_{mn}^3} - \frac{v_{lm}^a v_{mn}^b v_{nl}^c}{\epsilon_{ml}\epsilon_{mn}^3} \\ & - \frac{v_{ml}^a v_{nm}^b v_{ln}^c}{\epsilon_{ml}\epsilon_{mn}^3} - \frac{v_{lm}^a v_{nl}^b v_{mn}^c}{\epsilon_{ml}\epsilon_{mn}^3} \Big) - \frac{v_{nm}^{ac} v_{mn}^b}{\epsilon_{mn}^3} - \frac{v_{mn}^{ac} v_{nm}^b}{\epsilon_{mn}^3} - \frac{v_{nm}^{ab} v_{mn}^c}{\epsilon_{mn}^3} - \frac{v_{mn}^{ab} v_{nm}^c}{\epsilon_{mn}^3} \Big] \end{aligned} \tag{E-15}$$

Applying the key identity Off-diagonal Second Derivative, we collect all multiband contributions (denoted as $\mathcal{M}$) arising from: (1) the explicit expression of $T_0$, and (2) the expansion of the off-diagonal second derivative.

$$\begin{aligned} \mathcal{M} = \sum_{nml}' \frac{1}{2} f_n \Big[ \Big( & \frac{v_{ln}^a v_{nm}^b v_{ml}^c}{\epsilon_{ln}\epsilon_{mn}^3} + \frac{v_{ln}^a v_{nm}^c v_{ml}^b}{\epsilon_{ln}\epsilon_{mn}^3} + \frac{v_{nl}^a v_{lm}^c v_{mn}^b}{\epsilon_{ln}\epsilon_{mn}^3} + \frac{v_{nl}^a v_{lm}^b v_{mn}^c}{\epsilon_{ln}\epsilon_{mn}^3} \\ & - \frac{v_{ml}^a v_{ln}^b v_{nm}^c}{\epsilon_{ml}\epsilon_{mn}^3} - \frac{v_{lm}^a v_{mn}^b v_{nl}^c}{\epsilon_{ml}\epsilon_{mn}^3} - \frac{v_{ml}^a v_{nm}^b v_{ln}^c}{\epsilon_{ml}\epsilon_{mn}^3} - \frac{v_{lm}^a v_{nl}^b v_{mn}^c}{\epsilon_{ml}\epsilon_{mn}^3} \Big) \\ & + \Big( \frac{v_{nl}^a v_{lm}^c v_{mn}^b}{\epsilon_{nl}\epsilon_{mn}^3} - \frac{v_{nl}^c v_{lm}^a v_{mn}^b}{\epsilon_{lm}\epsilon_{mn}^3} + \frac{v_{ml}^a v_{ln}^c v_{nm}^b}{\epsilon_{ml}\epsilon_{mn}^3} - \frac{v_{ml}^c v_{ln}^a v_{nm}^b}{\epsilon_{ln}\epsilon_{mn}^3} \\ & + \frac{v_{nl}^a v_{lm}^b v_{mn}^c}{\epsilon_{nl}\epsilon_{mn}^3} - \frac{v_{nl}^b v_{lm}^a v_{mn}^c}{\epsilon_{lm}\epsilon_{mn}^3} + \frac{v_{ml}^a v_{ln}^b v_{nm}^c}{\epsilon_{ml}\epsilon_{mn}^3} - \frac{v_{ml}^b v_{ln}^a v_{nm}^c}{\epsilon_{ln}\epsilon_{mn}^3} \Big) \Big] = 0 \end{aligned} \tag{E-16}$$

Consequently, only terms involving two-band indices contribute:

$$\begin{aligned} T_0 = \sum_{nm}' \frac{1}{2} f_n \Big[ & 3\Delta_{mn}^a \frac{v_{nm}^b v_{mn}^c + v_{nm}^c v_{mn}^b}{\epsilon_{nm}^4} + \Delta_{mn}^b \frac{v_{nm}^a v_{mn}^c + v_{nm}^c v_{mn}^a}{\epsilon_{nm}^4} + \Delta_{mn}^c \frac{v_{nm}^a v_{mn}^b + v_{nm}^b v_{mn}^a}{\epsilon_{nm}^4} \\ & - \frac{1}{\epsilon_{mn}^3} \Big( -iR_{nm}^{ac} v_{nm}^c v_{mn}^b + \frac{v_{nm}^c v_{mn}^b \Delta_{nm}^a}{\epsilon_{nm}} + \frac{v_{nm}^a v_{mn}^b \Delta_{nm}^c}{\epsilon_{nm}} - iR_{mn}^{ac} v_{mn}^c v_{nm}^b + \frac{v_{mn}^c v_{nm}^b \Delta_{mn}^a}{\epsilon_{mn}} + \frac{v_{mn}^a v_{nm}^b \Delta_{mn}^c}{\epsilon_{mn}} \\ & - iR_{nm}^{ab} v_{nm}^b v_{mn}^c + \frac{v_{nm}^b v_{mn}^c \Delta_{nm}^a}{\epsilon_{nm}} + \frac{v_{nm}^a v_{mn}^c \Delta_{nm}^b}{\epsilon_{nm}} - iR_{mn}^{ab} v_{mn}^b v_{nm}^c + \frac{v_{mn}^b v_{nm}^c \Delta_{mn}^a}{\epsilon_{mn}} + \frac{v_{mn}^a v_{nm}^c \Delta_{mn}^b}{\epsilon_{mn}} \Big) \Big] \end{aligned} \tag{E-17}$$

We utilize the properties of shift vector $-i(R_{nm}^{ac} + R_{mn}^{ab}) v_{nm}^c v_{mn}^b = \epsilon_{nm}^2 \partial_a (A_{nm}^c A_{mn}^b)$:

$$T_0 = \sum_{nm}' \frac{1}{2} f_n \Big[ \frac{\partial_a (A_{nm}^c A_{mn}^b + A_{nm}^b A_{mn}^c)}{\epsilon_{nm}} + \Delta_{mn}^a \frac{v_{nm}^b v_{mn}^c + v_{nm}^c v_{mn}^b}{\epsilon_{nm}^4} \Big] \tag{E-18}$$

By further applying the Feynman-Hellmann identity and the definition of the band-renormalized quantum metric, the final expression is shown to be proportional to the derivative of the band-renormalized quantum metric:

$$\begin{aligned} T_0 &= \sum_{nm}' \frac{1}{2} f_n \Big[ \frac{\partial_a (A_{nm}^c A_{mn}^b + A_{nm}^b A_{mn}^c)}{\epsilon_{nm}} + (A_{nm}^b A_{mn}^c + A_{nm}^c A_{mn}^b) \partial_a (\frac{1}{\epsilon_{nm}}) \Big] \\ &= \sum_{nm}' \frac{1}{2} f_n \partial_a \Big( \frac{A_{nm}^c A_{mn}^b + A_{nm}^b A_{mn}^c}{\epsilon_{nm}} \Big) = \sum_n f_n \partial_a \mathcal{G}_n^{bc} \end{aligned} \tag{E-19}$$

### 5. Transformation and Final Expressions

We employ IBP in $k$-space to simplify the expressions and facilitate a comparison with the two-band model results presented in Supplementary Section D 4. Our derivation reveals that the final expressions of $\sigma_{abc}^{(0)}$ represent a natural generalization of the two-band model; the analytical form remains identical, requiring (1) the extension of the band summation index from $n \in \{1, 2\}$ to $n \in \{1, \ldots, N\}$ (2) replacing the two-band (Band-renormalized) Quantum Metric with their multiband counterpart. The detailed results of this derivation are summarized below:

#### *A. Kinetic Contribution*

The total intrinsic kinetic contribution is $\sigma_{abc}^{\mathrm{kin}} = T_4 + T_3$. By performing IBP on $T_3$ with respect to $k_a$, we obtain:

$$T_3 \xrightarrow{\mathrm{IBP}} -\sum\nolimits_n \frac{1}{2 \times 4!}(v_n^a v_n^c v_n^b) f_n^{(4)} = -\frac{1}{2} T_4 \tag{E-20}$$

Summing the resulting terms yields:

$$\sigma_{abc}^{\mathrm{kin}} = \frac{1}{2} T_4 = \sum\nolimits_n \frac{1}{48} v_n^a v_n^b v_n^c f_n^{(4)}, \quad f_n^{(4)} = 2 f_{0,n}^{(4)} \tag{E-21}$$

#### *B. Geometric Contribution*

The rest of intrinsic contribution is geometric $\sigma_{abc}^{\mathrm{geo}} = T_2 + T_1 + T_0$. First, we apply IBP to $T_0$:

$$T_0 \xrightarrow{\mathrm{IBP}} -\sum\nolimits_n f_n^{(1)} v_n^a \mathcal{G}_n^{bc} \tag{E-22}$$

Next, performing IBP on $T_2$ leads to:

$$T_2 \xrightarrow{\mathrm{IBP}} -\frac{1}{2} \sum\nolimits_n f_n^{(1)} \partial_a g_n^{bc} \tag{E-23}$$

Combining the above contributions with $T_1$ [Eq. (E-14)], we arrive at the final expression:

$$\begin{aligned} \sigma_{abc}^{\mathrm{geo}} &= -\frac{1}{2} \sum\nolimits_n f_n^{(1)} \partial_a g_n^{bc} + \sum\nolimits_n f_n^{(1)} \left[ v_n^b \mathcal{G}_n^{ac} + v_n^c \mathcal{G}_n^{ab} - v_n^a \mathcal{G}_n^{bc} \right] - \sum\nolimits_n f_n^{(1)} v_n^a \mathcal{G}_n^{bc} \\ &= \sum\nolimits_n f_n^{(1)} \left[ v_n^b \mathcal{G}_n^{ac} + v_n^c \mathcal{G}_n^{ab} - 2 v_n^a \mathcal{G}_n^{bc} - \frac{\partial_a g_n^{bc}}{2} \right], \quad f_n^{(1)} = 2 f_{0,n}^{(1)} \end{aligned} \tag{E-24}$$

This expression closely mirrors its two-band counterpart (Eq. D-71), yet entails a multiband generalization: the term $g_{ab}/\epsilon_{n\bar{n}}$ is now replaced by (or generalized to) the multiband band-renormalized quantum metric $\mathcal{G}_n^{ab}$.

## Appendix F: DC Conductivity from the Low-Frequency Limit of Second-Harmonic Generation (SHG)

In experimental practice, the low-frequency SHG conductivity, $\sigma_{abc}(-2\omega; \omega, \omega)$, is often preferred over the rectified conductivity, $\sigma_{abc}(\omega, -\omega)$. Typical experimental frequencies range from $10^1$ to $10^3$ Hz, which are significantly lower than the scattering rates in most materials. Consequently, these measurements can be effectively treated as the low-frequency limit ($\omega \to 0$) of the SHG response. To better align our theoretical framework with experimental observations, this section elucidates the SHG response in this limit.

We adopt a procedure analogous to the calculation of the rectified conductivity illustrated in Supplementary Section A, B and C: specifically, we evaluate the low-frequency limit of the response function to extract the frequency-independent nonlinear transport coefficients, and subsequently perform a relaxation-time expansion around $\Gamma = 0$:

$$\sigma_{abc}^{\mathrm{SHG}} = \frac{1}{\Gamma^2} \sigma_{abc}^{\mathrm{SHG}(-2)} + \frac{1}{\Gamma} \sigma_{abc}^{\mathrm{SHG}(-1)} + \sigma_{abc}^{\mathrm{SHG}(0)} + \mathcal{O}(\Gamma) \tag{F-1}$$

### 1. The $\mathcal{O}(\Gamma^{-2})$ (Nonlinear Drude) Contribution

The $\Gamma^{-2}$ contribution is given by:

$$\begin{aligned}\sigma_{abc}^{\mathrm{SHG}(-2)} &= \sum_n \frac{1}{2} f_n^{(2)} v_n^a v_n^b v_n^c + \frac{1}{8}\sum_n f_n^{(1)} \Big[ v_{nn}^{ac} v_n^b + v_{nn}^{ab} v_n^c + 4 v_{nn}^{bc} v_n^a \\ &+ \sum\nolimits_m' \Big( v_n^c \frac{v_{mn}^a v_{nm}^b + v_{mn}^b v_{nm}^a}{\epsilon_{nm}} + v_n^b \frac{v_{mn}^a v_{nm}^c + v_{mn}^c v_{nm}^a}{\epsilon_{nm}} + 4 v_n^a \frac{v_{mn}^b v_{nm}^c + v_{mn}^c v_{nm}^b}{\epsilon_{nm}} \Big) \Big]\end{aligned} \tag{F-2}$$

By applying the properties of the diagonal second derivative (as detailed in Supplementary Section E) and performing integration by parts (IBP), we obtain:

$$\begin{aligned}\sigma_{abc}^{\mathrm{SHG}(-2)} &= \sum_n \Big[ \frac{1}{2} f_n^{(2)} v_n^a v_n^b v_n^c + \frac{1}{8} f_n^{(1)} \Big( \partial_a \partial_c \epsilon_n v_n^b + \partial_a \partial_b \epsilon_n v_n^c + 4 \partial_b \partial_c \epsilon_n v_n^a \Big) \Big] \\ &\xrightarrow{\mathrm{IBP}} -\frac{1}{8} \sum_n f_n^{(1)} \partial_a (v_n^b v_n^c)\end{aligned} \tag{F-3}$$

Consequently, we find that in the low-frequency limit, the nonlinear Drude contribution to the SHG is identical to the corresponding expression for the rectified conductivity.

### 2. The $\mathcal{O}(\Gamma^{-1})$ (Berry Curvature Dipole) Contribution

The $\Gamma^{-1}$ contribution is expressed as:

$$\sigma_{abc}^{\mathrm{SHG}(-1)} = -\frac{i}{4} \sum\nolimits_{nm}' f_n^{(1)} \Big( v_n^c \frac{v_{mn}^a v_{nm}^b - v_{mn}^b v_{nm}^a}{\epsilon_{nm}^2} + v_n^b \frac{v_{mn}^a v_{nm}^c - v_{mn}^c v_{nm}^a}{\epsilon_{nm}^2} \Big) \tag{F-4}$$

Utilizing the multiband curvature-velocity relation:

$$\sigma_{abc}^{\mathrm{SHG}(-1)} = \frac{1}{4} \sum_n f_n^{(1)} \Big( v_n^c \Omega_n^{ab} + v_n^b \Omega_n^{ac} \Big) \tag{F-5}$$

From this result, it is evident that the $\Gamma^{-1}$ expansion of the SHG is directly related to the Berry curvature, and its expression is identical to that of the rectified conductivity.

### 3. The $\mathcal{O}(\Gamma^{0})$ (Intrinsic) Contribution

Analogous to the rectified conductivity, the SHG $\Gamma^0$ term $\sigma_{abc}^{\mathrm{SHG}(0)}$ is expanded in powers of the inverse temperature $\beta$ as follows:

$$\sigma_{abc}^{\mathrm{SHG}(0)} = T_0^{\mathrm{SHG}} + T_1^{\mathrm{SHG}} + T_2^{\mathrm{SHG}} + T_3^{\mathrm{SHG}} + T_4^{\mathrm{SHG}}. \tag{F-6}$$

*Term 4 ($\sim \beta^4$, $f^{(4)}$)*

The fourth-order SHG contribution ($T_4^{\mathrm{SHG}}$) is the negative of the one of the rectified current [$T_4$, Eq. (E-8)]:

$$T_4^{\mathrm{SHG}} = -T_4 = -\frac{1}{4!} \sum_n v_n^a v_n^b v_n^c f_n^{(4)} \tag{F-7}$$

*Term 3 ($\sim \beta^3$, $f^{(3)}$)*

The third-order contribution to the SHG response, $T_3^{\mathrm{SHG}}$, is expressed as:

$$\begin{aligned}T_3^{\mathrm{SHG}} &= -\frac{1}{2\times 4!} \sum_n f_n^{(3)} \Big[ v_{nn}^{ac} v_n^b + v_{nn}^{ab} v_n^c + 4 v_{nn}^{bc} v_n^a \\ &+ \sum\nolimits_m' \Big( v_n^c \frac{v_{nm}^a v_{mn}^b + v_{nm}^b v_{mn}^a}{\epsilon_{nm}} + v_n^b \frac{v_{nm}^a v_{mn}^c + v_{nm}^c v_{mn}^a}{\epsilon_{nm}} + 4 v_n^a \frac{v_{nm}^b v_{mn}^c + v_{nm}^c v_{mn}^b}{\epsilon_{nm}} \Big) \Big]\end{aligned} \tag{F-8}$$

By utilizing the properties of the Diagonal Second Derivative, we can simplify the expression to:

$$T_3^{\mathrm{SHG}} = -\frac{1}{2\times 4!}\sum_n f_n^{(3)}\left[\partial_a\partial_c\epsilon_n\partial_b\epsilon_n + \partial_a\partial_b\epsilon_n\partial_c\epsilon_n + 4\partial_b\partial_c\epsilon_n\partial_a\epsilon_n\right] \tag{F-9}$$

*Term 2 ($\sim \beta^2$, $f^{(2)}$)*

The second-order SHG term $T_2^{\mathrm{SHG}}$ is found to be identical to its counterpart in the rectified current:

$$T_2^{\mathrm{SHG}} = \frac{1}{2}\sum_n f_n^{(2)} v_n^a g_n^{bc} \tag{F-10}$$

*Term 1 ($\sim \beta^1$, $f^{(1)}$)*

The first-order SHG term $T_1^{\mathrm{SHG}}$ is given by:

$$T_1^{\mathrm{SHG}} = -\sum_{nm}' \frac{1}{2} f_n^{(1)}\left[v_n^a\frac{v_{nm}^b v_{mn}^c + v_{nm}^c v_{mn}^b}{\epsilon_{nm}^3} + v_n^b\frac{v_{nm}^a v_{mn}^c + v_{nm}^c v_{mn}^a}{\epsilon_{nm}^3} + v_n^c\frac{v_{nm}^a v_{mn}^b + v_{nm}^b v_{mn}^a}{\epsilon_{nm}^3}\right] \tag{F-11}$$

By invoking the multiband band-renormalized metric-velocity relation, this expression can be simplified as follows:

$$T_1^{\mathrm{SHG}} = -\sum_n f_n^{(1)}\left[v_n^b\mathcal{G}_n^{ac} + v_n^c\mathcal{G}_n^{ab} + v_n^a\mathcal{G}_n^{bc}\right] \tag{F-12}$$

*Term 0 ($\sim \beta^0$, $f^{(0)}$)*

The simplification of the SHG term $T_0^{\mathrm{SHG}}$ is the most analytically involved step. For clarity, we decompose the initial expression into five distinct components:

$$T_{0,1}^{\mathrm{SHG}} = \frac{1}{2}\sum_{nm}' f_n\left[\frac{\Delta_{nm}^a(v_{mn}^b v_{nm}^c + v_{mn}^c v_{nm}^b)}{\epsilon_{nm}^4} + 7\frac{\Delta_{nm}^c(v_{mn}^a v_{nm}^b + v_{mn}^b v_{nm}^a)}{\epsilon_{nm}^4} + 7\frac{\Delta_{nm}^b(v_{mn}^a v_{nm}^c + v_{mn}^c v_{nm}^a)}{\epsilon_{nm}^4}\right] \tag{F-13}$$

$$T_{0,2}^{\mathrm{SHG}} = -\frac{1}{2}\sum_{nm}' f_n\left(\frac{v_{nm}^{ac}v_{mn}^b}{\epsilon_{nm}^3} + \frac{v_{nm}^{ab}v_{mn}^c}{\epsilon_{nm}^3} + \frac{v_{mn}^{ac}v_{nm}^b}{\epsilon_{nm}^3} + \frac{v_{mn}^{ab}v_{nm}^c}{\epsilon_{nm}^3} + \frac{4v_{nm}^{bc}v_{mn}^a}{\epsilon_{nm}^3} + \frac{4v_{mn}^{bc}v_{nm}^a}{\epsilon_{nm}^3}\right) \tag{F-14}$$

$$\begin{aligned} T_{0,3}^{\mathrm{SHG}} = -\frac{1}{2}\sum_{nml}' f_n\Bigg(&\frac{v_{nl}^a v_{mn}^b v_{lm}^c}{\epsilon_{nl}\epsilon_{nm}^3} + \frac{v_{nl}^a v_{lm}^b v_{mn}^c}{\epsilon_{nl}\epsilon_{nm}^3} + \frac{v_{ln}^a v_{ml}^b v_{nm}^c}{\epsilon_{nl}\epsilon_{nm}^3} + \frac{v_{ln}^a v_{nm}^b v_{ml}^c}{\epsilon_{nl}\epsilon_{nm}^3} \\ &- \frac{v_{lm}^a v_{mn}^b v_{nl}^c}{\epsilon_{lm}\epsilon_{nm}^3} - \frac{v_{lm}^a v_{nl}^b v_{mn}^c}{\epsilon_{lm}\epsilon_{nm}^3} - \frac{v_{ml}^a v_{ln}^b v_{nm}^c}{\epsilon_{lm}\epsilon_{nm}^3} - \frac{v_{ml}^a v_{nm}^b v_{ln}^c}{\epsilon_{lm}\epsilon_{nm}^3}\Bigg) \end{aligned} \tag{F-15}$$

$$\begin{aligned} T_{0,4}^{\mathrm{SHG}} = -\sum_{nml}' f_n\Bigg(&\frac{v_{nl}^a v_{mn}^b v_{lm}^c}{\epsilon_{nl}^2\epsilon_{nm}^2} + \frac{v_{nl}^a v_{lm}^b v_{mn}^c}{\epsilon_{nl}^2\epsilon_{nm}^2} + \frac{v_{ln}^a v_{ml}^b v_{nm}^c}{\epsilon_{nl}^2\epsilon_{nm}^2} + \frac{v_{ln}^a v_{nm}^b v_{ml}^c}{\epsilon_{nl}^2\epsilon_{nm}^2} \\ &- \frac{v_{lm}^a v_{mn}^b v_{nl}^c}{\epsilon_{lm}^2\epsilon_{nm}^2} - \frac{v_{lm}^a v_{nl}^b v_{mn}^c}{\epsilon_{lm}^2\epsilon_{nm}^2} - \frac{v_{ml}^a v_{ln}^b v_{nm}^c}{\epsilon_{lm}^2\epsilon_{nm}^2} - \frac{v_{ml}^a v_{nm}^b v_{ln}^c}{\epsilon_{lm}^2\epsilon_{nm}^2}\Bigg) \end{aligned} \tag{F-16}$$

$$\begin{aligned} T_{0,5}^{\mathrm{SHG}} = -2\sum_{nml}' f_n\Bigg(&\frac{v_{nl}^a v_{mn}^b v_{lm}^c}{\epsilon_{nl}^3\epsilon_{nm}} + \frac{v_{nl}^a v_{lm}^b v_{mn}^c}{\epsilon_{nl}^3\epsilon_{nm}} + \frac{v_{ln}^a v_{ml}^b v_{nm}^c}{\epsilon_{nl}^3\epsilon_{nm}} + \frac{v_{ln}^a v_{nm}^b v_{ml}^c}{\epsilon_{nl}^3\epsilon_{nm}} \\ &- \frac{v_{lm}^a v_{mn}^b v_{nl}^c}{\epsilon_{lm}^3\epsilon_{ln}} - \frac{v_{lm}^a v_{nl}^b v_{mn}^c}{\epsilon_{lm}^3\epsilon_{ln}} - \frac{v_{ml}^a v_{ln}^b v_{nm}^c}{\epsilon_{lm}^3\epsilon_{ln}} - \frac{v_{ml}^a v_{nm}^b v_{ln}^c}{\epsilon_{lm}^3\epsilon_{ln}}\Bigg) \end{aligned} \tag{F-17}$$

The terms $T_{0,3}^{\mathrm{SHG}}$, $T_{0,4}^{\mathrm{SHG}}$, and $T_{0,5}^{\mathrm{SHG}}$ explicitly contain multiband contributions. Meanwhile, $T_{0,2}^{\mathrm{SHG}}$ involves the Off-diagonal Second Derivative; based on our analysis in Section E, expanding this term using the properties of Off-diagonal Second Derivatives will yield a series of additional multiband contributions.

Starting from $T^{\text{SHG}}_{0,2}$, we expand the Off-diagonal Second Derivatives first and decompose the shift vector into two distinct components: the logarithmic derivative $i\partial_a(\ln A^b_{nm})$ and the difference in intra-band Berry connections $(A^a_{nn} - A^a_{mm})$.

$$
\begin{aligned}
T^{\text{SHG}}_{0,2} = &-\frac{1}{2}{\sum}'_{nm}\frac{f_n}{\epsilon^3_{nm}}\Big[\partial_a(\ln A^c_{nm})v^c_{nm}v^b_{mn} + \partial_a(\ln A^b_{nm})v^b_{nm}v^c_{mn} + \partial_a(\ln A^c_{mn})v^c_{mn}v^b_{nm} + \partial_a(\ln A^b_{mn})v^b_{mn}v^c_{nm} \\
&+ 2\partial_c(\ln A^b_{nm})v^b_{nm}v^a_{mn} + 2\partial_c(\ln A^b_{mn})v^b_{mn}v^a_{nm} + 2\partial_b(\ln A^c_{nm})v^c_{nm}v^a_{mn} + 2\partial_b(\ln A^c_{mn})v^c_{mn}v^a_{nm} \\
&- 2i(A^c_{nn} - A^c_{mm})v^b_{nm}v^a_{mn} - 2i(A^c_{mm} - A^c_{nn})v^b_{mn}v^a_{nm} - 2i(A^b_{nn} - A^b_{mm})v^c_{nm}v^a_{mn} - 2i(A^b_{mm} - A^b_{nn})v^c_{mn}v^a_{nm} \\
&+ 2\frac{\Delta^a_{nm}(v^b_{mn}v^c_{nm} + v^c_{mn}v^b_{nm})}{\epsilon_{nm}} + 5\frac{\Delta^c_{nm}(v^a_{mn}v^b_{nm} + v^b_{mn}v^a_{nm})}{\epsilon_{nm}} + 5\frac{\Delta^b_{nm}(v^a_{mn}v^c_{nm} + v^c_{mn}v^a_{nm})}{\epsilon_{nm}} \\
&+ {\sum}'_l\Big( - 2\frac{v^b_{nl}v^c_{lm}v^a_{mn}}{\epsilon_{nl}} - 2\frac{v^c_{nl}v^b_{lm}v^a_{mn}}{\epsilon_{nl}} - 2\frac{v^c_{ml}v^b_{ln}v^a_{nm}}{\epsilon_{nl}} - 2\frac{v^b_{ml}v^c_{ln}v^a_{nm}}{\epsilon_{nl}} + 2\frac{v^c_{nl}v^b_{lm}v^a_{mn}}{\epsilon_{lm}} + 2\frac{v^b_{nl}v^c_{lm}v^a_{mn}}{\epsilon_{lm}} + 2\frac{v^b_{ml}v^c_{ln}v^a_{nm}}{\epsilon_{lm}} + 2\frac{v^c_{ml}v^b_{ln}v^a_{nm}}{\epsilon_{lm}} \\
&- \frac{v^a_{nl}v^b_{mn}v^c_{lm}}{\epsilon_{nl}} - \frac{v^a_{nl}v^b_{lm}v^c_{mn}}{\epsilon_{nl}} - \frac{v^a_{ln}v^b_{ml}v^c_{nm}}{\epsilon_{nl}} - \frac{v^a_{ln}v^b_{nm}v^c_{ml}}{\epsilon_{nl}} + \frac{v^a_{lm}v^b_{mn}v^c_{nl}}{\epsilon_{lm}} + \frac{v^a_{lm}v^b_{nl}v^c_{mn}}{\epsilon_{lm}} + \frac{v^a_{ml}v^b_{ln}v^c_{nm}}{\epsilon_{lm}} + \frac{v^a_{ml}v^b_{nm}v^c_{ln}}{\epsilon_{lm}}\Big)\Big]
\end{aligned}
\tag{F-18}
$$

We first notice the last two lines cancel $T^{\text{SHG}}_{0,3}$. Then we try to rewrite Berry-connection-difference part like $(A^c_{nn} - A^c_{mm})v^b_{nm}v^a_{mn}$ by using properties of off-diagonal second derivative again:

$$
\begin{aligned}
&- i(A^c_{mm} - A^c_{nn})v^b_{mn}v^a_{nm} - i(A^b_{nn} - A^b_{mm})v^c_{nm}v^a_{mn} \\
&= -\partial_a(\ln A^c_{nm})v^c_{nm}v^b_{mn} - \partial_a(\ln A^b_{mn})v^b_{mn}v^c_{nm} + \partial_c(\ln A^a_{nm})v^a_{nm}v^b_{mn} + \partial_b(\ln A^a_{mn})v^a_{mn}v^c_{nm} \\
&+ {\sum}'_l\Big( - \frac{v^c_{nl}v^a_{lm}v^b_{mn}}{\epsilon_{nl}} + \frac{v^a_{nl}v^c_{lm}v^b_{mn}}{\epsilon_{lm}} - \frac{v^b_{ml}v^a_{ln}v^c_{nm}}{\epsilon_{ml}} + \frac{v^a_{ml}v^b_{ln}v^c_{nm}}{\epsilon_{ln}} + \frac{v^a_{nl}v^c_{lm}v^b_{mn}}{\epsilon_{nl}} - \frac{v^c_{nl}v^a_{lm}v^b_{mn}}{\epsilon_{lm}} + \frac{v^a_{ml}v^b_{ln}v^c_{nm}}{\epsilon_{ml}} - \frac{v^b_{ml}v^a_{ln}v^c_{nm}}{\epsilon_{ln}}\Big)
\end{aligned}
\tag{F-19}
$$

By combining $T^{\text{SHG}}_{0,2}$ and $T^{\text{SHG}}_{0,3}$, we can decompose the resulting expression into a two-band term, $T^{\text{SHG; two-band}}_{0,2+3}$, and a multiband contribution, $T^{\text{SHG; multiband}}_{0,2+3}$:

$$
\begin{aligned}
T^{\text{SHG}}_{0,2+3} &= T^{\text{SHG}}_{0,2} + T^{\text{SHG}}_{0,3} \\
&= -\frac{1}{2}{\sum}'_{nm}\frac{f_n}{\epsilon^3_{nm}}\Big[\partial_a(\ln A^c_{nm})v^c_{nm}v^b_{mn} + \partial_a(\ln A^b_{nm})v^b_{nm}v^c_{mn} + \partial_a(\ln A^c_{mn})v^c_{mn}v^b_{nm} + \partial_a(\ln A^b_{mn})v^b_{mn}v^c_{nm} \\
&+ 2\partial_c(\ln A^b_{nm})v^b_{nm}v^a_{mn} + 2\partial_c(\ln A^b_{mn})v^b_{mn}v^a_{nm} + 2\partial_b(\ln A^c_{nm})v^c_{nm}v^a_{mn} + 2\partial_b(\ln A^c_{mn})v^c_{mn}v^a_{nm} \\
&+ 2\partial_c(\ln A^a_{mn})v^a_{mn}v^b_{nm} + 2\partial_c(\ln A^a_{nm})v^a_{nm}v^b_{mn} + 2\partial_b(\ln A^a_{mn})v^a_{mn}v^c_{nm} + 2\partial_b(\ln A^a_{nm})v^a_{nm}v^c_{mn} \\
&- 2\partial_a(\ln A^c_{nm})v^c_{nm}v^b_{mn} - 2\partial_a(\ln A^b_{nm})v^b_{nm}v^c_{mn} - 2\partial_a(\ln A^c_{mn})v^c_{mn}v^b_{nm} - 2\partial_a(\ln A^b_{mn})v^b_{mn}v^c_{nm} \\
&+ 2\frac{\Delta^a_{nm}(v^b_{mn}v^c_{nm} + v^c_{mn}v^b_{nm})}{\epsilon_{nm}} + 5\frac{\Delta^c_{nm}(v^a_{mn}v^b_{nm} + v^b_{mn}v^a_{nm})}{\epsilon_{nm}} + 5\frac{\Delta^b_{nm}(v^a_{mn}v^c_{nm} + v^c_{mn}v^a_{nm})}{\epsilon_{nm}} \\
&+ {\sum}'_l\Big( - 2\frac{v^b_{nl}v^c_{lm}v^a_{mn}}{\epsilon_{nl}} - 2\frac{v^c_{nl}v^b_{lm}v^a_{mn}}{\epsilon_{nl}} - 2\frac{v^c_{ml}v^b_{ln}v^a_{nm}}{\epsilon_{nl}} - 2\frac{v^b_{ml}v^c_{ln}v^a_{nm}}{\epsilon_{nl}} \\
&+ 2\frac{v^c_{nl}v^b_{lm}v^a_{mn}}{\epsilon_{lm}} + 2\frac{v^b_{nl}v^c_{lm}v^a_{mn}}{\epsilon_{lm}} + 2\frac{v^b_{ml}v^c_{ln}v^a_{nm}}{\epsilon_{lm}} + 2\frac{v^c_{ml}v^b_{ln}v^a_{nm}}{\epsilon_{lm}} \\
&+ 2\frac{v^a_{ml}v^c_{ln}v^b_{nm}}{\epsilon_{ml}} + 2\frac{v^a_{ml}v^b_{ln}v^c_{nm}}{\epsilon_{ml}} + 2\frac{v^a_{nl}v^c_{lm}v^b_{mn}}{\epsilon_{nl}} + 2\frac{v^a_{nl}v^b_{lm}v^c_{mn}}{\epsilon_{nl}} \\
&- 2\frac{v^c_{ml}v^a_{ln}v^b_{nm}}{\epsilon_{ln}} - 2\frac{v^b_{ml}v^a_{ln}v^c_{nm}}{\epsilon_{ln}} - 2\frac{v^c_{nl}v^a_{lm}v^b_{mn}}{\epsilon_{lm}} - 2\frac{v^b_{nl}v^a_{lm}v^c_{mn}}{\epsilon_{lm}} \\
&- 2\frac{v^c_{ml}v^a_{ln}v^b_{nm}}{\epsilon_{ml}} - 2\frac{v^b_{ml}v^a_{ln}v^c_{nm}}{\epsilon_{ml}} - 2\frac{v^c_{nl}v^a_{lm}v^b_{mn}}{\epsilon_{nl}} - 2\frac{v^b_{nl}v^a_{lm}v^c_{mn}}{\epsilon_{nl}} \\
&+ 2\frac{v^a_{ml}v^c_{ln}v^b_{nm}}{\epsilon_{ln}} + 2\frac{v^a_{ml}v^b_{ln}v^c_{nm}}{\epsilon_{ln}} + 2\frac{v^a_{nl}v^c_{lm}v^b_{mn}}{\epsilon_{lm}} + 2\frac{v^a_{nl}v^b_{lm}v^c_{mn}}{\epsilon_{lm}}\Big)\Big] = T^{\text{SHG; two-band}}_{0,2+3} + T^{\text{SHG; multiband}}_{0,2+3}
\end{aligned}
\tag{F-20}
$$

At this stage, we shall now group the multiband contributions:

$$
\begin{aligned}
\mathcal{M} &= T_{0,2+3}^{\mathrm{SHG;\ multiband}} + T_{0,4}^{\mathrm{SHG}} + T_{0,5}^{\mathrm{SHG}} \\
&= -\frac{1}{2}\sum\nolimits'_{nml} f_n \Big[ \frac{2v_{nl}^a v_{mn}^b v_{lm}^c}{\epsilon_{nl}^2\epsilon_{nm}^2} + \frac{2v_{nl}^a v_{lm}^b v_{mn}^c}{\epsilon_{nl}^2\epsilon_{nm}^2} + \frac{2v_{ln}^a v_{ml}^b v_{nm}^c}{\epsilon_{nl}^2\epsilon_{nm}^2} + \frac{2v_{ln}^a v_{nm}^b v_{ml}^c}{\epsilon_{nl}^2\epsilon_{nm}^2} \\
&- \frac{2v_{lm}^a v_{mn}^b v_{nl}^c}{\epsilon_{lm}^2\epsilon_{nm}^2} - \frac{2v_{lm}^a v_{nl}^b v_{mn}^c}{\epsilon_{lm}^2\epsilon_{nm}^2} - \frac{2v_{ml}^a v_{ln}^b v_{nm}^c}{\epsilon_{lm}^2\epsilon_{nm}^2} - \frac{2v_{ml}^a v_{nm}^b v_{ln}^c}{\epsilon_{lm}^2\epsilon_{nm}^2} \\
&+ \frac{4v_{nl}^a v_{mn}^b v_{lm}^c}{\epsilon_{nl}^3\epsilon_{nm}} + \frac{4v_{nl}^a v_{lm}^b v_{mn}^c}{\epsilon_{nl}^3\epsilon_{nm}} + \frac{4v_{ln}^a v_{ml}^b v_{nm}^c}{\epsilon_{nl}^3\epsilon_{nm}} + \frac{4v_{ln}^a v_{nm}^b v_{ml}^c}{\epsilon_{nl}^3\epsilon_{nm}} \\
&- \frac{4v_{lm}^a v_{mn}^b v_{nl}^c}{\epsilon_{lm}^3\epsilon_{ln}} - \frac{4v_{lm}^a v_{nl}^b v_{mn}^c}{\epsilon_{lm}^3\epsilon_{ln}} - \frac{4v_{ml}^a v_{ln}^b v_{nm}^c}{\epsilon_{lm}^3\epsilon_{ln}} - \frac{4v_{ml}^a v_{nm}^b v_{ln}^c}{\epsilon_{lm}^3\epsilon_{ln}} \\
&- 2\frac{v_{nl}^b v_{lm}^c v_{mn}^a}{\epsilon_{nl}\epsilon_{nm}^3} - 2\frac{v_{nl}^c v_{lm}^b v_{mn}^a}{\epsilon_{nl}\epsilon_{nm}^3} - 2\frac{v_{ml}^c v_{ln}^b v_{nm}^a}{\epsilon_{nl}\epsilon_{nm}^3} - 2\frac{v_{ml}^b v_{ln}^c v_{nm}^a}{\epsilon_{nl}\epsilon_{nm}^3} \\
&+ 2\frac{v_{nl}^c v_{lm}^b v_{mn}^a}{\epsilon_{lm}\epsilon_{nm}^3} + 2\frac{v_{nl}^b v_{lm}^c v_{mn}^a}{\epsilon_{lm}\epsilon_{nm}^3} + 2\frac{v_{ml}^b v_{ln}^c v_{nm}^a}{\epsilon_{lm}\epsilon_{nm}^3} + 2\frac{v_{ml}^c v_{ln}^b v_{nm}^a}{\epsilon_{lm}\epsilon_{nm}^3} \\
&+ \frac{2}{\epsilon_{nm}^2\epsilon_{ml}\epsilon_{ln}}\Big(v_{nl}^a v_{lm}^c v_{mn}^b + v_{nl}^a v_{lm}^b v_{mn}^c + v_{ml}^c v_{ln}^a v_{nm}^b + v_{ml}^b v_{ln}^a v_{nm}^c\Big) \\
&- \frac{2}{\epsilon_{nm}^2\epsilon_{ml}\epsilon_{ln}}\Big(v_{ml}^a v_{ln}^c v_{nm}^b + v_{nl}^c v_{lm}^a v_{mn}^b + v_{ml}^a v_{ln}^b v_{nm}^c + v_{nl}^b v_{lm}^a v_{mn}^c\Big)\Big]
\end{aligned}
\tag{F-21}
$$

Next, we reclassify these multiband contributions according to their matrix element products, appropriately interchanging the dummy indices $m \leftrightarrow l$ where necessary. Upon summing the coefficients of these matrix products, we find that the total multiband contribution vanishes identically.

$$
\begin{aligned}
\mathcal{M} = \frac{1}{2}\sum\nolimits'_{nml} f_n \Big[&(2v_{lm}^a v_{mn}^b v_{nl}^c + 2v_{lm}^a v_{mn}^c v_{nl}^b)\Big(\frac{1}{\epsilon_{lm}^2\epsilon_{nm}^2} + \frac{1}{\epsilon_{lm}^2\epsilon_{nl}^2} + \frac{2}{\epsilon_{lm}^3\epsilon_{ln}} - \frac{2}{\epsilon_{ml}^3\epsilon_{nm}} + \frac{1}{\epsilon_{nm}^2\epsilon_{ml}\epsilon_{ln}} + \frac{1}{\epsilon_{nl}^2\epsilon_{lm}\epsilon_{mn}}\Big) \\
&+ (v_{ln}^a v_{nm}^b v_{ml}^c + v_{ln}^a v_{nm}^c v_{ml}^b + v_{nl}^a v_{lm}^b v_{mn}^c + v_{nl}^a v_{lm}^c v_{mn}^b)\Big(-\frac{2}{\epsilon_{nl}^2\epsilon_{nm}^2} - \frac{4}{\epsilon_{nl}^3\epsilon_{nm}} + \frac{2}{\epsilon_{nl}^3\epsilon_{nm}} - \frac{2}{\epsilon_{nl}^3\epsilon_{ml}} - \frac{2}{\epsilon_{nm}^2\epsilon_{ml}\epsilon_{ln}}\Big)\Big] \\
= 0 &
\end{aligned}
\tag{F-22}
$$

The vanishing of $\mathcal{M}$ signifies that multiband effects do not introduce any contributions to the final response. This is consistent with the case of the rectified current. Consequently, $T_0^{\mathrm{SHG}}$ reduces to a combination of two-band terms:

$$
\begin{aligned}
T_0^{\mathrm{SHG}} &= T_{0,1}^{\mathrm{SHG}} + T_{0,2+3}^{\mathrm{SHG;\ two\text{-}band}} \\
&= -\frac{1}{2}\sum\nolimits'_{nm}\frac{f_n}{\epsilon_{nm}^3}\Big[\partial_a(\ln A_{nm}^c)v_{nm}^c v_{mn}^b + \partial_a(\ln A_{nm}^b)v_{nm}^b v_{mn}^c + \partial_a(\ln A_{mn}^c)v_{mn}^c v_{nm}^b + \partial_a(\ln A_{mn}^b)v_{mn}^b v_{nm}^c \\
&+ 2\partial_c(\ln A_{nm}^b)v_{nm}^b v_{mn}^a + 2\partial_c(\ln A_{mn}^b)v_{mn}^b v_{nm}^a + 2\partial_b(\ln A_{nm}^c)v_{nm}^c v_{mn}^a + 2\partial_b(\ln A_{mn}^c)v_{mn}^c v_{nm}^a \\
&+ 2\partial_c(\ln A_{mn}^a)v_{mn}^a v_{nm}^b + 2\partial_c(\ln A_{nm}^a)v_{nm}^a v_{mn}^b + 2\partial_b(\ln A_{mn}^a)v_{mn}^a v_{nm}^c + 2\partial_b(\ln A_{nm}^a)v_{nm}^a v_{mn}^c \\
&- 2\partial_a(\ln A_{nm}^c)v_{nm}^c v_{mn}^b - 2\partial_a(\ln A_{nm}^b)v_{nm}^b v_{mn}^c - 2\partial_a(\ln A_{mn}^c)v_{mn}^c v_{nm}^b - 2\partial_a(\ln A_{mn}^b)v_{mn}^b v_{nm}^c \\
&+ 2\frac{\Delta_{nm}^a(v_{mn}^b v_{nm}^c + v_{mn}^c v_{nm}^b)}{\epsilon_{nm}} + 5\frac{\Delta_{nm}^c(v_{mn}^a v_{nm}^b + v_{mn}^b v_{nm}^a)}{\epsilon_{nm}} + 5\frac{\Delta_{nm}^b(v_{mn}^a v_{nm}^c + v_{mn}^c v_{nm}^a)}{\epsilon_{nm}} \\
&- \frac{\Delta_{nm}^a(v_{mn}^b v_{nm}^c + v_{mn}^c v_{nm}^b)}{\epsilon_{nm}} - 7\frac{\Delta_{nm}^c(v_{mn}^a v_{nm}^b + v_{mn}^b v_{nm}^a)}{\epsilon_{nm}} - 7\frac{\Delta_{nm}^b(v_{mn}^a v_{nm}^c + v_{mn}^c v_{nm}^a)}{\epsilon_{nm}}\Big]
\end{aligned}
\tag{F-23}
$$

By invoking the Feynman-Hellmann identity, this expression can be further transformed into:

$$
\begin{aligned}
T_0^{\mathrm{SHG}} = \frac{1}{2}\sum\nolimits'_{nm} f_n \Big[&\frac{\partial_a(A_{nm}^c A_{mn}^b + A_{nm}^b A_{mn}^c)}{\epsilon_{nm}} - 2\frac{\partial_b(A_{nm}^a A_{mn}^c + A_{nm}^c A_{mn}^a)}{\epsilon_{nm}} - 2\frac{\partial_c(A_{nm}^a A_{mn}^b + A_{nm}^b A_{mn}^a)}{\epsilon_{nm}} \\
&+(A_{nm}^c A_{mn}^b + A_{nm}^b A_{mn}^c)\partial_a(\frac{1}{\epsilon_{nm}}) - 2(A_{nm}^a A_{mn}^c + A_{nm}^c A_{mn}^a)\partial_b(\frac{1}{\epsilon_{nm}}) - 2(A_{nm}^a A_{mn}^b + A_{nm}^b A_{mn}^a)\partial_c(\frac{1}{\epsilon_{nm}})\Big]
\end{aligned}
\tag{F-24}
$$

Finally, we observe that these partial derivatives can be reorganized into the form of total derivatives of the band-renormalized quantum metric:

$$
T_0^{\mathrm{SHG}} = -\sum\nolimits_n f_n(2\partial_c\mathcal{G}_n^{ab} + 2\partial_b\mathcal{G}_n^{ac} - \partial_a\mathcal{G}_n^{bc})
\tag{F-25}
$$

### 4. Transformation and Final Expressions

Upon evaluating the low-frequency SHG conductivity, several intermediate terms appear formally distinct from those of the rectified conductivity. However, once summarized and categorized using a similar scheme, low-frequency SHG and rectified conductivities are revealed to be identical. Here we employ IBP to simplify the expressions.

#### *A. Kinetic Contribution*

The total intrinsic kinetic contribution is $\sigma_{abc}^{\text{SHG;kin}} = T_4 + T_3$. For $T_3$, we utilize the relation $4\partial_b\partial_c\epsilon_n\partial_a\epsilon_n = -2\partial_a(\partial_b\epsilon_n\partial_c\epsilon_n) + 2\partial_b(\partial_a\epsilon_n\partial_c\epsilon_n) + 2\partial_c(\partial_a\epsilon_n\partial_b\epsilon_n)$ and perform IBP:

$$\begin{aligned} T_3^{\text{SHG}} &= -\frac{1}{2\times 4!}\sum\nolimits_n f_n^{(3)}\Big[\partial_a\partial_c\epsilon_n\partial_b\epsilon_n + \partial_a\partial_b\epsilon_n\partial_c\epsilon_n + 4\partial_b\partial_c\epsilon_n\partial_a\epsilon_n\Big] \\ &= -\frac{1}{2\times 4!}\sum\nolimits_n f_n^{(3)}\Big[-\partial_a(\partial_b\epsilon_n\partial_c\epsilon_n) + 2\partial_b(\partial_a\epsilon_n\partial_c\epsilon_n) + 2\partial_c(\partial_a\epsilon_n\partial_b\epsilon_n)\Big] \xrightarrow{\text{IBP}} \frac{3}{2\times 4!}\sum\nolimits_n f_n^{(4)} v_n^a v_n^b v_n^c \end{aligned} \tag{F-26}$$

Summing these contributions, we arrive at the final expression:

$$\begin{aligned} \sigma_{abc}^{\text{SHG;kin}} &= -\frac{1}{4!}\sum\nolimits_n v_n^a v_n^b v_n^c f_n^{(4)} + \frac{3}{2\times 4!}\sum\nolimits_n f_n^{(4)} v_n^a v_n^b v_n^c \\ &= \frac{1}{48}\sum\nolimits_n v_n^a v_n^b v_n^c f_n^{(4)}, \quad f_n^{(4)} = 2f_{0,n}^{(4)} \end{aligned} \tag{F-27}$$

#### *B. Geometric Contribution*

The rest of intrinsic contribution is geometric $\sigma_{abc}^{\text{SHG;geo}} = T_2 + T_1 + T_0$. By performing integration by parts (IBP) on $T_0^{\text{SHG}}$, we can recast it into a form analogous to $T_1$:

$$T_0^{\text{SHG}} \xrightarrow{\text{IBP}} \sum\nolimits_n f_n^{(1)}(2v_n^c\mathcal{G}_n^{ab} + 2v_n^b\mathcal{G}_n^{ac} - v_n^a\mathcal{G}_n^{bc}) \tag{F-28}$$

Likewise, performing IBP on $T_2$ yields:

$$T_2^{\text{SHG}} \xrightarrow{\text{IBP}} -\frac{1}{2}\sum\nolimits_n f_n^{(1)}\partial_a g_n^{bc} \tag{F-29}$$

Summing these three components, we obtain the total geometric contribution:

$$\begin{aligned} \sigma_{abc}^{\text{SHG;geo}} &= -\frac{1}{2}\sum\nolimits_n f_n^{(1)}\partial_a g_n^{bc} - \sum\nolimits_n f_n^{(1)}\Big[v_n^b\mathcal{G}_n^{ac} + v_n^c\mathcal{G}_n^{ab} + v_n^a\mathcal{G}_n^{bc}\Big] + \sum\nolimits_n f_n^{(1)}(2v_n^c\mathcal{G}_n^{ab} + 2v_n^b\mathcal{G}_n^{ac} - v_n^a\mathcal{G}_n^{bc}) \\ &= \sum\nolimits_n f_n^{(1)}\Big[v_n^b\mathcal{G}_n^{ac} + v_n^c\mathcal{G}_n^{ab} - 2v_n^a\mathcal{G}_n^{bc} - \frac{\partial_a g_n^{bc}}{2}\Big], \quad f_n^{(1)} = 2f_{0,n}^{(1)} \end{aligned} \tag{F-30}$$

## Appendix G: Non-equivalence of the Fermionic Bath Model to the RTA/IFR Framework

Although our results of the $\Gamma^0$ nonlinear conductivity for the wide-band fermionic bath share certain characteristics with the RTA/IFR framework, the discrepancies in coefficients and the emergence of extra contributions demonstrate that the two approaches are inherently non-equivalent. To substantiate this, we show that reducing our exactly solvable model to the phenomenological RTA/IFR form requires exceptionally crude approximations.

First, we derive the exact equation of motion for the reduced density matrix. Following the approach by Matsyshyn et al. [17], when considering a fermionic bath, the reduced density matrix can be expressed in the following form:

$$\rho_S(t) = \sum_{n,j} f_0(\varepsilon_j)\,|\psi_n^j(t)\rangle\,\langle\psi_n^j(t)| \tag{G-1}$$

where $|\psi_n^j(t)\rangle$ is the component within the system Hilbert space evolving from the initial state of the fermionic bath $|\phi_{n,j}\rangle$. The exact equation of motion for $|\psi_n^j(t)\rangle$ is given by:

$$i\partial_t\,|\psi_n^j(t)\rangle = [H_S(t) - i\Gamma]\,|\psi_n^j(t)\rangle + \lambda\exp[-i\varepsilon_j(t-t_0)]\,|\chi_n\rangle\,. \tag{G-2}$$

The above equation is a simple non-Hermitian open-system Schrödinger equation, in which the system Hamiltonian is dressed by a constant imaginary part "$-i\Gamma$" which captures the decay into the bath. However, we see that the bath is not only to induce decay, but also produces the source term $\lambda \exp[-i\varepsilon_j(t-t_0)]\,|\chi_n\rangle$ that makes the equation inhomogeneous. The balance of decay and source term is what allows the existence of nontrivial non-equilibrium steady state.

From this, we derive the equation of motion for the reduced density matrix:

$$\begin{aligned} i\partial_t\rho_S(t) &= i\sum_{n,j} f_0(\varepsilon_j)\Big[\Big(\partial_t\,|\psi_n^j(t)\rangle\Big)\langle\psi_n^j(t)| + |\psi_n^j(t)\rangle\Big(\partial_t\,\langle\psi_n^j(t)|\Big)\Big] \\ &= \sum_{n,j} f_0(\varepsilon_j)\Big[(H_S(t)-i\Gamma)\,|\psi_n^j(t)\rangle\langle\psi_n^j(t)| + \lambda\exp[-i\varepsilon_j(t-t_0)]\,|\chi_n\rangle\langle\psi_n^j(t)|\Big] \\ &- \sum_{n,j} f_0(\varepsilon_j)\Big[\,|\psi_n^j(t)\rangle\langle\psi_n^j(t)|\,(H_S(t)+i\Gamma) + \lambda\exp[i\varepsilon_j(t-t_0)]\,|\psi_n^j(t)\rangle\langle\chi_n|\Big] \\ &= [H_S(t),\rho_S(t)] - 2i\Gamma\rho_S(t) + \lambda\sum_{n,j} f_0(\varepsilon_j)\Big[\exp[-i\varepsilon_j(t-t_0)]\,|\chi_n\rangle\langle\psi_n^j(t)| - \text{h.c.}\Big] \\ &\equiv [H_S(t),\rho_S(t)] - 2i\Gamma\rho_S(t) + \mathcal{S}[\psi_n^j(t)] \end{aligned} \tag{G-3}$$

Again we see a source term $\mathcal{S}[\psi_n^j(t)]$ that makes the equation inhomogeneous. We examine the source term $\mathcal{S}[\psi_n^j(t)]$. To force this exact dynamic equation to reduce to the standard RTA/IFR form, one needs to apply two approximations:

First, one needs to apply a *static Hamiltonian approximation* to the source term. Specifically, one must artificially freeze the system Hamiltonian within the equations of motion, enforcing $H_S(t) \to H_S(0)$. This entirely discards the dynamic dressing of the states by the driving field. Under this assumption, a direct calculation yields:

$$|\psi_n^j(t)\rangle = -\frac{\lambda}{\epsilon_n - i\Gamma - \varepsilon_j} e^{-i\varepsilon_j(t-t_0)}\,|\chi_n\rangle \tag{G-4}$$

$$\begin{aligned} \mathcal{S}[\psi_n^j(t)] &= \sum_{n,j} f_0(\varepsilon_j)\frac{2i\lambda^2\Gamma}{(\epsilon_n-\varepsilon_j)^2+\Gamma^2}\,|\chi_n\rangle\langle\chi_n| = \sum_n \int_{-\infty}^{\infty} d\omega_b f_0(\omega_b)\frac{2i\lambda^2\Gamma\sum_j\delta(\varepsilon_j-\omega_b)}{(\epsilon_n-\omega_b)^2+\Gamma^2}\,|\chi_n\rangle\langle\chi_n| \\ &= 2i\Gamma\sum_n\int_{-\infty}^{\infty}\frac{d\omega_b}{\pi} f_0(\omega_b)\frac{\Gamma}{(\epsilon_n-\omega_b)^2+\Gamma^2}\,|\chi_n\rangle\langle\chi_n| \end{aligned} \tag{G-5}$$

Here, we have employed the assumption of a featureless fermionic bath, characterized by the spectral density $\nu_B = \sum_j 2\pi\delta(\varepsilon_j - \omega_b)$, along with the definition of the relaxation rate $\Gamma = \lambda^2\nu_B/2$.

Second, one needs to further perform a *relaxation rate truncation*. By expanding the dissipation parameter around $\Gamma = 0$ and strictly retaining only the first-order term, one effectively replaces the exact Lorentzian spectral broadening with a Dirac delta function:

$$\begin{aligned} \mathcal{S}[\psi_n^j(t)] &= 2i\Gamma\sum_n\int_{-\infty}^{\infty}\frac{d\omega_b}{\pi} f_0(\omega_b)\frac{\Gamma}{(\epsilon_n-\omega_b)^2+\Gamma^2}\,|\chi_n\rangle\langle\chi_n| \\ &= 2i\Gamma\sum_n f_0(\epsilon_n)\,|\chi_n\rangle\langle\chi_n| + O(\Gamma^2) = 2i\Gamma\rho_0 + O(\Gamma^2) \end{aligned} \tag{G-6}$$

We observe that $\mathcal{S}[\psi_n^j(t)]$ reduces to $2i\Gamma\rho_0 + O(\Gamma^2)$, which is proportional to the density matrix of the fermionic system at equilibrium when all higher-order $\Gamma$ contributions are ignored. These higher-order contributions can be systematically extracted by expressing the integral as a convolution and expanding its Fourier-space representation via the Taylor series of $e^{-\Gamma|k|}$:

$$\int \frac{d\omega_b}{\pi} f_0(\omega_b)\,\frac{\Gamma}{(\epsilon_n-\omega_b)^2+\Gamma^2} = \frac{1}{2\pi}\int dk\,\tilde{f}_0(k)e^{ik\epsilon_n}e^{-\Gamma|k|} \tag{G-7}$$

Applying these two approximations reduces the source term to $\mathcal{S}[\psi_n^j(t)] \simeq 2i\Gamma\rho_0$. Defining the relaxation time $\tau = (2\Gamma)^{-1}$, the equation of motion for the reduced density matrix simplifies to the phenomenological RTA form:

$$\partial_t\rho_S(t) = -i[H_S(t),\rho_S(t)] - \frac{\rho_S(t)-\rho_0}{\tau} \tag{G-8}$$

This reproduces the quantum Liouville equation used by RTA/IFR framework [39]. However, as previously discussed, these approximations are crude for nonlinear transport. The first approximation artificially deletes all Floquet information at finite AC frequencies, while the second truncation ignores all higher-order $O(\Gamma^2)$ contributions generated by the drive and the bath. Consequently, our exact Markovian fermionic bath model cannot be reduced to the RTA/IFR framework which is often employed to describe impurity scatterings without discarding the specific mechanisms that actively shape the observable intrinsic conductivity.