Editors' Suggestion

Dissipation-Shaped Quantum Geometry in Nonlinear Transport

Zhichao Guo, Xing-Yuan Liu, Hua Wang, Li-kun Shi,* and Kai Chang†
Center for Quantum Matter, School of Physics, Zhejiang University, Hangzhou 310058, China

(Received 2 December 2025; revised 9 March 2026; accepted 20 April 2026; published 18 May 2026)

The theory of the intrinsic nonlinear Hall effect, a key probe of quantum geometry, is plagued by conflicting expressions for the conductivity that is independent of the dissipation *strength* (rate, Γ^0). We clarify the origin of this ambiguity by demonstrating that the "intrinsic" response is not universal, but is inextricably linked to how the dissipation *mechanism* shapes the nonequilibrium steady state (NESS) density matrix. We establish a benchmark by solving the exact NESS density matrix for a generic Bloch system coupled to a featureless fermionic bath. Our exact Γ^0 conductivity decomposes into two parts: (i) a geometric contribution, σ^{geo}, which establishes the definitive structure of the quantum metric contribution (including the intraband $\sim\partial_k g$ term), clarifying inconsistencies in the literature, and (ii) a novel, purely kinetic contribution, $\sigma^{\mathrm{kin}} \propto v^3 f_0^{(4)}$, arising from mechanism-specific modifications to the occupation functions, which is absent in approaches that postulate, rather than derive, the relaxation dynamics. The discrepancies in both σ^{geo} and σ^{kin} between these distinct physical mechanisms prove that the Γ^0 nonlinear conductivity is not a unique property of the Bloch Hamiltonian, but is contingent on the physical system-bath coupling.

DOI: 10.1103/rbck-mt1k

Introduction—Linear response theory establishes a profound connection between transport phenomena and the properties of the thermodynamic equilibrium state, which are insensitive to the specific nature of the environment, as exemplified by the relationship between the intrinsic anomalous Hall effect and the Berry curvature [1–7]. This paradigm, however, breaks down in the nonlinear regime. DC nonlinear responses are governed not by equilibrium properties, but by the nonequilibrium steady state (NESS). The structure of the NESS, i.e., the nonequilibrium density matrix, emerges from a delicate balance between the external drive and dissipation.

Importantly, we must distinguish between the *mechanism* of dissipation (the microscopic nature of the system-environment coupling, e.g., metallic backgate, phonons, or disorder statistics) and the *strength* of dissipation (the phenomenological rate, Γ). The mechanism determines the analytical structure of the NESS density matrix (e.g., how occupation functions are modified) [8–20], and is therefore central, not peripheral, to the resulting nonlinear transport.

Recent years have witnessed intense theoretical efforts to interpret nonlinear transport through the lens of quantum geometry [21–56]. Many of these efforts aim to identify "intrinsic" contributions, which are independent of dissipation strength (Γ^0) and are characterized by the quantum metric [1,2,7,21,22,25,30,35,38,53,54]. Despite significant progress, the field is plagued by inconsistencies. Different theoretical frameworks, including semiclassical wave-packet dynamics [23,33,34,51], quantum kinetics or Green's function methods [26–28,31,32,36,37,39,40,46,47,50,52], yield conflicting expressions for these "intrinsic" conductivities.

We argue that these inconsistencies expose a fundamental physical misconception. The implicit assumption that the Γ^0 conductivity is universal, defined solely by the system Hamiltonian, is flawed. The NESS, which governs DC transport, is inextricably linked to the specific physical *mechanism* of dissipation (Fig. 1). The analytical structure

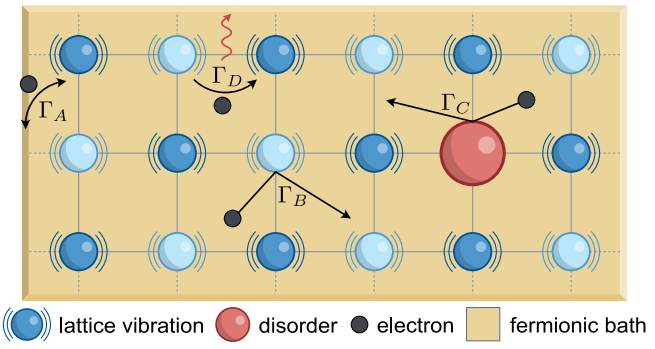

FIG. 1. Nonuniversality of the Γ^0 nonlinear conductivity. The non-equilibrium density matrix governing DC transport depends on the specific dissipation *mechanism*, leading to mechanism-dependent "intrinsic" (Γ^0) conductivities. Illustrated environments mediating relaxation: (Γ_A) fermionic bath (e.g., metallic backgate); (Γ_B) electron-phonon interactions; (Γ_C) static disorder; (Γ_D) radiative processes.

*Contact author: likun.shi@zju.edu.cn
†Contact author: kchang@zju.edu.cn

TABLE I. Comparison of the "intrinsic" (Γ^0) contributions to the second-order DC conductivity $\sigma^{(0)}_{abc} = \sigma^{\mathrm{geo}}_{abc} + \sigma^{\mathrm{kin}}_{abc}$ derived from different theoretical approaches and the dissipation mechanisms they implicitly or explicitly model. The discrepancies in both $\sigma^{\mathrm{geo}}_{abc}$ (varying coefficients c_1/c_2 and the intraband term $\partial_a g^{bc}_n$) and $\sigma^{\mathrm{kin}}_{abc}$ between the distinct dissipation mechanisms demonstrate the nonuniversality of the Γ^0 response. Here, g^{ab}_n (G^{ab}_n) is the (energy normalized) quantum metric for band n, v^a_n is the band velocity, $f'_{0,n}$ and $f^{(4)}_{0,n}$ are the first and fourth derivatives of the equilibrium Fermi-Dirac distribution f_0. Different conventions for symmetrizing σ_{abc} (or electron charge) can lead to an overall factor of 2 (or -1) difference.

Methods	Dissipation mechanism	Γ^0 Geometric $\sigma^{\mathrm{geo}}_{abc}$	Γ^0 Kinetic $\sigma^{\mathrm{kin}}_{abc}$
Wavepacket or Liouville dynamics with RTAs or IFRs (e.g., Refs. [23,39,47,51])	Phenomenological c_1/c_2 varies depending on different assumptions	$\sum_{n,\mathbf{k}} \left[c_1(G^{ab}_n v^c_n + G^{ac}_n v^b_n) - c_2 G^{bc}_n v^a_n\right] f'_{0,n}$	0
Green's function [52]	Constant-Γ self-energy	$\sum_{n,\mathbf{k}} \left[(G^{ab}_n v^c_n + G^{ac}_n v^b_n - 2G^{bc}_n v^a_n) - (\partial_a g^{bc}_n/2)\right] f'_{0,n}$	0
Open-system Schrödinger equation (this work)	Microscopic: fermionic bath	$\sum_{n,\mathbf{k}} \left[(G^{ab}_n v^c_n + G^{ac}_n v^b_n - 2G^{bc}_n v^a_n) - (\partial_a g^{bc}_n/2)\right] 2f'_{0,n}$	$\sum_{n,\mathbf{k}} (v^a_n v^b_n v^c_n)(f^{(4)}_{0,n}/24)$

of the nonequilibrium density matrix, imposed by the mechanism dictates the precise procedure for taking the DC limit ($\omega \to 0$) followed by the Γ expansion.

The reliance on phenomenological treatments in the literature obscures this fundamental dependence. Methods such as relaxation time approximations (RTAs) and imaginary frequency regularizations (IFRs) do not derive the analytical structure of the nonequilibrium density matrix from microscopic principles; instead, they impose it via *ad hoc* assumptions (e.g., regarding how occupations relax). The resulting Γ^0 conductivity therefore depends on these choices. For instance, RTA implementations differ on the postulated physical target state for relaxation: some assume relaxation towards the equilibrium distribution of the unperturbed Hamiltonian, $f_0(\epsilon)$ [47], while others assume relaxation toward a quasiequilibrium state defined by the field-modified energy (e.g., incorporating the Stark shift), $f_0(\tilde{\epsilon})$ [23,51].

Further ambiguity arises in the regularization procedure required for the DC limit; e.g., there is no consensus on whether the regularization factor should remain constant, $i\Gamma$ [31,32], or depend on the order of the perturbation, $iN\Gamma$ [39,46,47,50], sometimes rationalized as a mathematical device for adiabatic switching. These technical differences change how divergences are regularized when taking the $\omega \to 0$ followed by Γ expansions. This proliferation of approaches, based on distinct phenomenological assumptions, leads to the conflicting expressions summarized in Table I (varying c_1/c_2), underscoring the inadequacy of theories that do not specify the physical dissipation mechanism.

In this Letter, we provide a theoretical benchmark by analyzing a concrete, microscopically defined open quantum system: a crystalline electronic system coupled to a wide-band fermionic bath. While idealized, this model allows for a controlled derivation of the NESS, capturing the essential physics of particle exchange with a Markovian fermionic environment (e.g., metallic back gate) [17,19,20,41,57–59] and providing a necessary contrast to idealized phenomenological models.

Our analysis yields the definitive second-order conductivity σ_{abc} for this specific mechanism. We find that the Γ^0 contribution contains not only a specific form of the quantum metric contribution, comprising both interband and intraband (derivative) terms, but also a novel, Γ^0 kinetic term proportional to the fourth derivative of the Fermi function, $f^{(4)}_0$. The specific analytical form of the geometric term and the appearance of the kinetic term demonstrate that the structure of the Γ^0 nonlinear conductivity is nonuniversal and depends on the nature of the system-bath coupling.

Microscopic open-system formalism—To avoid the ambiguities inherent in phenomenological treatments of dissipation, we analyze a concrete open quantum system where the regularization of the NESS is microscopically determined, rather than postulated. The total Hamiltonian we consider is $H(t) = H_S(t) + H_B + H_{SB}$. The system, $H_S(t)$, describes Bloch electrons driven by a spatially uniform electric field $\mathbf{E}(t)$. We employ the velocity gauge, incorporating the drive via the Peierls substitution (setting $e = \hbar = 1$) $H_S(t) = H_0[\mathbf{k} - \mathbf{A}(t)]$, where $H_0 = h_{mn}(\mathbf{k})|\chi_m\rangle\langle\chi_n|$ is the unperturbed Bloch Hamiltonian with eigenstates $|\chi_n\rangle$ (incorporating both momentum and band indices) and $\mathbf{E}(t) = -\partial_t \mathbf{A}(t)$. The environment is modeled as a fermionic bath [17,41,57,59]. We adopt the bath Hamiltonian $H_B = \sum_{n,j} \epsilon_j |\varphi_{n,j}\rangle\langle\varphi_{n,j}|$ and the system-bath tunnel coupling $H_{SB} = \lambda \sum_{n,j} (|\chi_n\rangle\langle\varphi_{n,j}| + \mathrm{H.c.})$. This Bloch eigenstate coupling structure describes a translationally invariant interaction, enforced by momentum conservation when the system couples uniformly to the bath. This model is physically motivated by van der Waals

heterostructures; for instance, when a 2D electronic system couples uniformly to a metallic backgate (e.g., graphite) via an ultrathin tunnel barrier (e.g., ∼2 nm hBN). This model provides an elementary theoretical benchmark, distinct from spatially localized coupling (e.g., contacts at system edges), and enables a direct comparison with translationally invariant phenomenological models and a clear formulation of transport theory in terms of momentum-space quantum geometry.

The bath is initialized in thermal equilibrium characterized by the Fermi-Dirac distribution $f_0(\epsilon)=[e^{\beta(\epsilon-\mu)}+1]^{-1}$ with $1/\beta$ the bath temperature and μ the bath chemical potential. We employ the wideband approximation, treating the bath's density of states as featureless; i.e., $\nu_B(\omega)=\nu_0$. This model is in the same class of those noninteracting fermionic models often described within the Keldysh formalism [26,59–64], and captures the essential physics of a Markovian environment and yields a well-defined, energy-independent relaxation rate $\Gamma=\lambda^2\nu_0/2$.

The dynamics of the system's reduced density matrix, $\rho_S(t)$, are determined by tracing out the bath degrees of freedom [19,41]. To extract the nonlinear conductivity, we expand the system's reduced density matrix perturbatively with respect to the driving $\rho_S(t)=\rho^{(0)}+\rho^{(1)}(t)+\rho^{(2)}(t)+\mathcal{O}(V^3)$, where $V(t)=H_S(t)-H_0$. The microscopic derivation yields exact expressions for the response kernels (detailed in Supplemental Material [65]). Their analytical structure differ significantly from those obtained via RTAs or IFRs.

Crucially, this specific structure determines how the fermionic bath regularizes divergences when taking the DC ($\omega\to 0$) limit followed by Γ expansion. In RTAs, DC corrections to the distribution function typically diverge as powers of $1/\Gamma$. In contrast, the fermionic bath regularization leads to *finite*, Γ^0 corrections to the NESS occupation functions even in the clean limit. This distinction highlights a fundamental principle: the precise analytical form of these Γ-independent corrections is inherently determined by the physical dissipation mechanism. It is this mechanism-specific structure of the NESS in the clean limit that constitutes the origin of the nonuniversal Γ^0 conductivity [19,20].

The DC nonlinear conductivity—By applying the microscopic formalism to a generic multiband system coupled to the fermionic bath, we compute the second-order DC conductivity σ_{abc} [65]. This result is derived directly from the exact NESS density matrix, obtained by solving the open-system dynamics without resorting to phenomenological regularizations. This microscopic derivation yields the precise analytical structure of the NESS, which dictates the subsequent expansion of the resulting conductivity by powers of $1/\Gamma$:

$$\sigma_{abc}=\frac{1}{\Gamma^2}\sigma^{(-2)}_{abc}+\frac{1}{\Gamma}\sigma^{(-1)}_{abc}+\sigma^{(0)}_{abc}+\mathcal{O}(\Gamma). \tag{1}$$

We present the results using the Fermi-Dirac distribution derivatives $f^{(k)}_{0,n}=\partial^k f_0(\epsilon)/\partial\epsilon^k|_{\epsilon_n}$ (with $f'_{0,n}\equiv f^{(1)}_{0,n}$ following literature conventions), along with the conventional definition of the quantum metric $g^{ab}_n=\sum_{m\neq n}\mathrm{Re}[A^a_{nm}A^b_{mn}]$ [21,22], and Berry curvature $\Omega^n_{ab}=-2\sum_{m\neq n}\mathrm{Im}[A^a_{nm}A^b_{mn}]$ [1,2], where $A^a_{nm}=\langle n|i\partial_a|m\rangle$ represents the Berry connection and $\partial_a\equiv\partial/\partial k_a$.

(1) Γ-dependent contributions $(\Gamma^{-2},\Gamma^{-1})$: the leading divergence, $\sigma^{(-2)}_{abc}$, corresponds to the nonlinear Drude response,

$$\sigma^{(-2)}_{abc}=-\frac{1}{4}\sum_{n,\mathbf{k}}\partial_a(v^b_n v^c_n)f'_{0,n}, \tag{2}$$

while the $\mathcal{O}(\Gamma^{-1})$ term corresponds to the nonlinear Hall effect (NAHE), related to the Berry curvature Ω^n_{ab},

$$\sigma^{(-1)}_{abc}=\frac{1}{2}\sum_{n,\mathbf{k}}(v^c_n\Omega^n_{ab}+v^b_n\Omega^n_{ac})f'_{0,n}. \tag{3}$$

This recovers the well-known expression for the Berry curvature dipole (BCD) contribution [24,29,41], serving as a validation of the formalism.

(2) Γ^0 contributions: the Γ-independent term, $\sigma^{(0)}_{abc}$, is the central result of this work and the focus of discrepancies in the literature. The open-system formalism reveals a decomposition into distinct kinetic and geometric components:

$$\sigma^{(0)}_{abc}=\sigma^{\mathrm{kin}}_{abc}+\sigma^{\mathrm{geo}}_{abc}. \tag{4}$$

First, we identify a novel purely kinetic contribution:

$$\sigma^{\mathrm{kin}}_{abc}=\sum_{n,\mathbf{k}}\frac{1}{24}v^a_n v^b_n v^c_n f^{(4)}_{0,n}. \tag{5}$$

This term represents a Γ-independent, ballistic nonlinear response. Its emergence is a direct consequence of the specific NESS regularization imposed by the fermionic bath, which leads to finite, Γ^0 corrections to the NESS occupation functions persisting in the clean limit ($\Gamma\to 0$). These corrections manifest analytically through high-order Polygamma functions [65], which generate the high-order energy derivative $f^{(4)}_0$, even in single-band models without Berry phases [17]. This regularization is physically distinct from that of RTAs, explaining why $\sigma^{\mathrm{kin}}_{abc}$ is absent in those models. This contribution is maximized at low temperatures or when sharp band structure features exist near the Fermi level.

Second, we obtain the contributions arising from the quantum geometry:

$$\sigma^{\mathrm{geo}}_{abc}=\sum_{n,\mathbf{k}}\left[(G^{ab}_n v^c_n+G^{ac}_n v^b_n-2G^{bc}_n v^a_n)-\frac{\partial_a g^{bc}_n}{2}\right]2f'_{0,n}. \tag{6}$$

This expression provides the definitive quantum metric contribution for a system coupled to the featureless

fermionic bath. It separates into two physically distinct parts, both originating from field-induced geometric shifts of the electron wave packets. The first term represents the energy-normalized interband quantum metric $G_n^{ab} \equiv \sum_{m\neq n} \mathrm{Re}[A^a_{nm}A^b_{mn}]/(\epsilon_n - \epsilon_m)$ [23,39,47]. The second term, $-\partial_a g_{bc}/2$ is the intraband quantum metric contribution [52]. Physically, it describes the contribution to the current arising from the deformation of the wave packet characterized by the metric g_{bc}, which quantifies the wave packet's spatial spread. It emerges explicitly when geometric contributions involving f'' are transformed to the Fermi surface (f') via integration by parts [65].

The results summarized in Eqs. (5) and (6) provide the definitive analytical proof that the Γ^0 conductivity is nonuniversal. Both the kinetic contribution $\sigma^{\mathrm{kin}}_{abc}$ and the specific structure of $\sigma^{\mathrm{geo}}_{abc}$, including the intraband term $(-\partial_a g_{bc}/2)$ and the exact prefactors of the interband terms, are characteristic of the fermionic bath. The discrepancies with RTA models, which exhibit varying geometric prefactors and lack the kinetic term (Table I), demonstrate that $\sigma^{(0)}_{abc}$ cannot be viewed as a property of the isolated system.

Illustrative model calculation—To demonstrate our central findings, the nonuniversality of $\sigma^{(0)}_{abc}$ and the physical significance of the novel kinetic term $\sigma^{\mathrm{kin}}_{abc}$, we apply our microscopic formalism to a concrete lattice model [51]. We consider a 2D $\mathcal{PT}$-symmetric metal on a square lattice (lattice constant a) described by the 4×4 tight-binding Hamiltonian

$$\mathcal{H}(\mathbf{k}) = v\tau_x[\sin(k_x a)\sigma_x + \sin(k_y a)\sigma_y] + M(\mathbf{k})\tau_z, \quad (7)$$

where $M(\mathbf{k}) = m + b[\cos(k_x a) + \cos(k_y a)] + t\sin(k_x a)$. This Hamiltonian is invariant under the combined antiunitary symmetry $\mathcal{PT}$ (with $\mathcal{P} = \tau_z$, $\mathcal{T} = i\sigma_y\mathcal{K}$) but explicitly breaks both the inversion symmetry $\mathcal{P}$ and the time-reversal symmetry $\mathcal{T}$ via the $t\sin(k_x a)$ term, making it an ideal testbed for second-order DC responses without BCD contributions [23,34].

The Hamiltonian [Eq. (7)] commutes with the symmetry operator $S = \tau_z\sigma_z$. This allows for a unitary transformation that separates $\mathcal{H}(\mathbf{k})$ into the $S = \pm 1$ eigenspaces, yielding the block-diagonal form

$$\mathcal{H}'(\mathbf{k}) = \begin{bmatrix} h(\mathbf{k}) & 0 \\ 0 & h^*(\mathbf{k}) \end{bmatrix}, \quad h(\mathbf{k}) = \begin{pmatrix} M_{\mathbf{k}} & v_- \\ v_+ & -M_{\mathbf{k}} \end{pmatrix}, \quad (8)$$

with $v_\pm = v[\sin(k_x a) \pm i\sin(k_y a)]$. This structure simplifies the analysis by reducing the original 4×4 Hamiltonian to two copies of two-band system. We numerically evaluate the two competing Γ^0 contributions for this model: the novel kinetic term $\sigma^{\mathrm{kin}}_{xyy}$ [Eq. (5)] and the geometric term $\sigma^{\mathrm{geo}}_{xyy}$ [Eq. (6)]. The results are presented in Fig. 2.

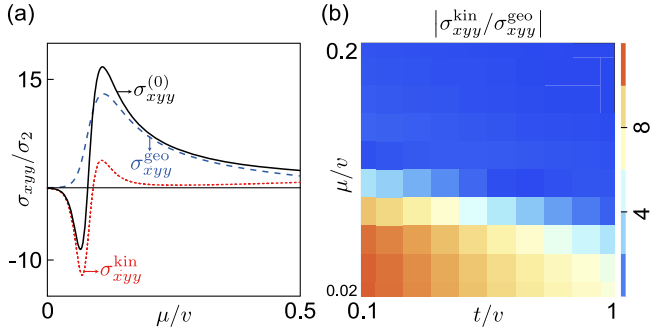

FIG. 2. Competition between Γ^0 geometric and kinetic nonlinear conductivity in the $\mathcal{PT}$-symmetric model [Eq. (7)]. (a) Calculated Γ^0 contributions $\sigma^{\mathrm{geo}}_{xyy}$ (blue dashed) and $\sigma^{\mathrm{kin}}_{xyy}$ (red dotted), along with the total Γ^0 response $\sigma^{(0)}_{xyy}$ (black solid), as a function of chemical potential μ. The kinetic term is comparable in magnitude to the geometric term. [Parameters used: $b/v = 1$, $m/v = -3.9$, $t/v = 0.9$, $\beta/v = 100$; characteristic second order conductivity $\sigma_2 = a(\hbar/v)\cdot e^3/\hbar^2$]. (b) Color map of the ratio $|\sigma^{\mathrm{kin}}_{xyy}/\sigma^{\mathrm{geo}}_{xyy}|$ as a function of tilt parameter t and chemical potential μ. The kinetic term [Eq. (5)] dominates (red regions) over the geometric term [Eq. (6)] in significant portions of the parameter space, underscoring its non-negligible role in the Γ^0 nonlinear response when the system is coupled to a fermionic bath.

Figure 2(a) compares the chemical potential (μ) dependence of $\sigma^{\mathrm{kin}}_{xyy}$ and $\sigma^{\mathrm{geo}}_{xyy}$ for fixed model parameters. The geometric term, scaling as $\sim f_0'$, is a Fermi surface contribution that peaks near the band edges. The kinetic term, $\sigma^{\mathrm{kin}}_{xyy} \propto f_0^{(4)}$, exhibits a more complex, oscillatory dependence on μ. Due to the high-order derivative, $\sigma^{\mathrm{kin}}_{xyy}$ is strongly sensitive to the sharpness of the band features near the Fermi level and is enhanced at low temperatures.

Figure 2(b) maps the ratio of the two contributions, $|\sigma^{\mathrm{kin}}_{xyy}/\sigma^{\mathrm{geo}}_{xyy}|$, as a function of the tilt parameter t and chemical potential μ. This demonstrates that $\sigma^{\mathrm{kin}}_{xyy}$ is not merely a minor correction; it becomes comparable to, or even dominant over, the geometric term across large regions of the parameter space, particularly where the band curvature is significant.

This calculation demonstrates the quantitative significance of the mechanism-dependent kinetic term. While theories based on RTAs predict a *purely* geometric Γ^0 response (see Table I), the open-system analysis shows that the fermionic bath establishes an NESS where $\sigma^{\mathrm{kin}}_{abc}$ is a necessary and often comparable component of the total conductivity.

Discussion and conclusion—We have established that the "intrinsic" Γ^0 nonlinear conductivity depends on the dissipation *mechanism*. By analyzing a Bloch system coupled to a microscopic fermionic bath, we derived an exact result for the second-order DC conductivity, σ_{abc}, providing two key insights.

First, we clarified the structure of the Γ^0 geometric contribution, $\sigma^{\mathrm{geo}}_{abc}$ [Eq. (6)], confirming the presence of

both intraband $(\sim\partial_a g_{bc})$ and specific interband QMD terms. This result agrees with constant-Γ Green's function formalisms [52], suggesting $\sigma^{\mathrm{geo}}_{abc}$ may be characteristic for idealized Markovian models. The contrast with the conflicting results from RTA-based theories [39,47,51] evidences that the dissipation mechanism shapes the quantum geometric response.

Second, we uncovered a novel, Γ^0 kinetic contribution, $\sigma^{\mathrm{kin}}_{abc} \sim v^3 f_0^{(4)}$ [Eq. (5)]. This term emerges from the specific way the fermionic bath modifies the electronic occupation functions, which persists even in the clean limit. The emergence of $\sigma^{\mathrm{kin}}_{abc}$ as a finite, mechanism-specific response, distinct from simplified phenomenological models, evidences this dependence. In fact, this dependence on the dissipation mechanism extends beyond DC responses and quantum geometry. For example, analysis of AC-driven Floquet systems shows that the rectification current differs when coupled to a bosonic versus a fermionic bath, even in a single Bloch band without quantum geometry and in the weak-coupling limit [20].

Our open-system formulation of the dissipation-shaped NESS offers a new perspective on recent experiments and theories regarding the nonlinear responses in $\mathcal{PT}$ symmetric antiferromagnets, specifically even-layered $MnBi_2Te_4$ [42,43,45]. These experiments extracted a Γ-independent intercept via scaling laws, operationally defining it as the intrinsic nonlinear response. However, Wang *et al.* [42] measured a finite longitudinal response in pristine devices on bare substrates, while Gao *et al.* [43,45] reported a predominantly transverse response using encapsulated van der Waals heterostructures. Our findings offer a physical approach to reconcile these distinct observations by examining both the tensor structure of the response and the underlying dissipation mechanisms.

Analytically, the geometric response is not strictly restricted to be transverse. Within phenomenological RTA models [51,53], the longitudinal geometric response vanishes due to the strict cancellation $(G^{aa}v^a + G^{aa}v^a - 2G^{aa}v^a = 0)$. However, in agreement with Ulrich *et al.* [52], our geometric response shaped by the fermionic bath contains an intraband quantum metric term, $-\partial_a g_{bc}/2$ [Eq. (6)]. Because this yields a nonzero term for the longitudinal direction $(-\partial_a g_{aa}/2)$, it demonstrates that an intrinsic longitudinal geometric response is physically allowed.

This emergence of the intraband metric term is just one manifestation of a broader principle: the operationally extracted Γ^0 nonlinear response is not a universal band structure property, but is actively shaped by the specific dissipation mechanism. Extending beyond the geometric tensor structure, solving the exact dynamics for a system coupled to a wide-band fermionic bath shows the resulting nonlinear dc conductivity also includes a kinetic contribution [Eq. (5)]. Because the aforementioned experiments utilized different architectures, ranging from bare substrates [42] to encapsulated heterostructures [43,45], the electrons coupled to distinct baths. Consequently, the variations in the extracted longitudinal-to-transverse ratios are physically expected: different dissipation mechanisms drive the system to distinct NESS, thereby generating different observable nonlinear transport signatures.

This principle of nonuniversality also clarifies ongoing theoretical efforts to define the intrinsic nonlinear current [53,54]. Specifically, Xiao *et al.* [53] demonstrated that a DC electric field shifts the local wavepacket energy, driving the system to a modified steady-state distribution. We support this overall physical picture but emphasize that the precise distribution is nonuniversal. Separately, Resta [54] formulated the geometric positional shift within an exact many-body framework for an isolated system. Moving beyond this coherent limit, our approach emphasizes the importance of specific dissipation mechanisms in actively shaping the observable steady-state DC response.

This work reframes the understanding of "intrinsic" nonlinear response, emphasizing a departure from the linear response paradigm. The environment is not merely parameterized by Γ; the specific dissipation *mechanism* actively shapes the nonequilibrium steady state density matrix, determining the precise occupation functions and coherences, and consequently, the observable conductivity. The resulting nonuniversality indicates that a single, universal expression for the "intrinsic" conductivity is untenable. By showing that quantum geometry in nonlinear transport is shaped by dissipation, this work invites investigation into how specific, experimentally relevant physical environments influence nonlinear transport signatures.

Acknowledgments—We would like to thank Yannis Ulrich for helpful discussions. This work is supported by the National Natural Science Foundation of China (NSFC; Grants No. 12488101 and No. 92265203), the Strategic Priority Research Program of the Chinese Academy of Sciences (Grants No. XDB0460000 and No. XDB28000000), and the Quantum Science and Technology-National Science and Technology Major Project (Grants No. 2024ZD0300104 and No. 2021ZD0302600). H. W. acknowledges the support from the NSFC under Grants No. 12522411 and No. 12474240.

Data availability—The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

[1] D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, Quantized Hall conductance in a two-dimensional periodic potential, Phys. Rev. Lett. **49**, 405 (1982).

[2] M. V. Berry, Quantal phase factors accompanying adiabatic changes, Proc. R. Soc. A **392**, 45 (1984).

[3] F. D. M. Haldane, Model for a quantum Hall effect without Landau levels: Condensed-matter realization of the "parity anomaly," Phys. Rev. Lett. **61**, 2015 (1988).
[4] N. Sinitsyn, Semiclassical theories of the anomalous Hall effect, J. Phys. Condens. Matter **20**, 023201 (2007).
[5] N. Nagaosa, J. Sinova, S. Onoda, A. H. MacDonald, and N. P. Ong, Anomalous Hall effect, Rev. Mod. Phys. **82**, 1539 (2010).
[6] D. Xiao, M.-C. Chang, and Q. Niu, Berry phase effects on electronic properties, Rev. Mod. Phys. **82**, 1959 (2010).
[7] R. Resta, The insulating state of matter: A geometrical theory, Eur. Phys. J. B **79**, 121 (2011).
[8] P. Tien and J. Gordon, Multiphoton process observed in the interaction of microwave fields with the tunneling between superconductor films, Phys. Rev. **129**, 647 (1963).
[9] R. G. Mani, J. H. Smet, K. von Klitzing, V. Narayanamurti, W. B. Johnson, and V. Umansky, Zero-resistance states induced by electromagnetic-wave excitation in GaAs/AlGaAs heterostructures, Nature (London) **420**, 646 (2002).
[10] M. Zudov, R. Du, L. Pfeiffer, and K. West, Evidence for a new dissipationless effect in 2D electronic transport, Phys. Rev. Lett. **90**, 046807 (2003).
[11] M. Bukov, L. D'Alessio, and A. Polkovnikov, Universal high-frequency behavior of periodically driven systems: From dynamical stabilization to Floquet engineering, Adv. Phys. **64**, 139 (2015).
[12] M. Sentef, Light-enhanced electron-phonon coupling from nonlinear electron-phonon coupling, Phys. Rev. B **95**, 205111 (2017).
[13] T. Oka and S. Kitamura, Floquet engineering of quantum materials, Annu. Rev. Condens. Matter Phys. **10**, 387 (2019).
[14] R. Kumari, B. Seradjeh, and A. Kundu, Josephson-current signatures of unpaired Floquet majorana fermions, Phys. Rev. Lett. **133**, 196601 (2024).
[15] T. Le, R. Jiang, L. Tu, R. Bian, Y. Ma, Y. Shi, K. Jia, Z. Li, Z. Lyu, X. Cao *et al.*, Inverse-current quantum electro-oscillations in a charge density wave insulator, Phys. Rev. B **109**, 245123 (2024).
[16] C. J. Eckhardt, S. Chattopadhyay, D. M. Kennes, E. A. Demler, M. A. Sentef, and M. H. Michael, Theory of resonantly enhanced photo-induced superconductivity, Nat. Commun. **15**, 2300 (2024).
[17] O. Matsyshyn, J. C. Song, I. S. Villadiego, and L.-k. Shi, Fermi-Dirac staircase occupation of Floquet bands and current rectification inside the optical gap of metals: An exact approach, Phys. Rev. B **107**, 195135 (2023).
[18] P. Zhu and A. Alexandradinata, Anomalous shift and optical vorticity in the steady photovoltaic current, Phys. Rev. B **110**, 115108 (2024).
[19] L.-k. Shi, O. Matsyshyn, J. C. Song, and I. S. Villadiego, Floquet Fermi liquid, Phys. Rev. Lett. **132**, 146402 (2024).
[20] L.-k. Shi, O. Matsyshyn, J. C. Song, and I. S. Villadiego, Ultracritical Floquet non-Fermi liquid, Phys. Rev. Lett. **134**, 196401 (2025).
[21] J. P. Provost and G. Vallee, Riemannian structure on manifolds of quantum states, Commun. Math. Phys. **76**, 289 (1980).
[22] A. K. Pati, Relation between "phases" and "distance" in quantum evolution, Phys. Lett. A **159**, 105 (1991).
[23] Y. Gao, S. A. Yang, and Q. Niu, Field induced positional shift of Bloch electrons and its dynamical implications, Phys. Rev. Lett. **112**, 166601 (2014).
[24] I. Sodemann and L. Fu, Quantum nonlinear Hall effect induced by Berry curvature dipole in time-reversal invariant materials, Phys. Rev. Lett. **115**, 216806 (2015).
[25] S. Peotta and P. Törmä, Superfluidity in topologically nontrivial flat bands, Nat. Commun. **6**, 8944 (2015).
[26] N. Nagaosa and T. Morimoto, Concept of quantum geometry in optoelectronic processes in solids: Application to solar cells, Adv. Mater. **29**, 1603345 (2017).
[27] H. Wang and X. Qian, Ferroelectric nonlinear anomalous Hall effect in few-layer WTe2, npj Comput. Mater. **5**, 119 (2019).
[28] O. Matsyshyn and I. Sodemann, Nonlinear Hall acceleration and the quantum rectification sum rule, Phys. Rev. Lett. **123**, 246602 (2019).
[29] Q. Ma, S.-Y. Xu, H. Shen, D. MacNeill, V. Fatemi, T.-R. Chang, A. M. Mier Valdivia, S. Wu, Z. Du, C.-H. Hsu *et al.*, Observation of the nonlinear Hall effect under time-reversal-symmetric conditions, Nature (London) **565**, 337 (2019).
[30] F. Xie, Z. Song, B. Lian, and B. A. Bernevig, Topology-bounded superfluid weight in twisted bilayer graphene, Phys. Rev. Lett. **124**, 167002 (2020).
[31] H. Watanabe and Y. Yanase, Nonlinear electric transport in odd-parity magnetic multipole systems: Application to mn-based compounds, Phys. Rev. Res. **2**, 043081 (2020).
[32] H. Watanabe and Y. Yanase, Chiral photocurrent in parity-violating magnet and enhanced response in topological antiferromagnet, Phys. Rev. X **11**, 011001 (2021).
[33] C. Wang, Y. Gao, and D. Xiao, Intrinsic nonlinear Hall effect in antiferromagnetic tetragonal CuMnAs, Phys. Rev. Lett. **127**, 277201 (2021).
[34] H. Liu, J. Zhao, Y.-X. Huang, W. Wu, X.-L. Sheng, C. Xiao, and S. A. Yang, Intrinsic second-order anomalous Hall effect and its application in compensated antiferromagnets, Phys. Rev. Lett. **127**, 277202 (2021).
[35] J. Cayssol and J.-N. Fuchs, Topological and geometrical aspects of band theory, J. Phys. **4**, 034007 (2021).
[36] Z. Du, C. Wang, H.-P. Sun, H.-Z. Lu, and X. Xie, Quantum theory of the nonlinear Hall effect, Nat. Commun. **12**, 5038 (2021).
[37] R. Oiwa and H. Kusunose, Systematic analysis method for nonlinear response tensors, J. Phys. Soc. Jpn. **91**, 014701 (2022).
[38] P. Törmä, Essay: Where can quantum geometry lead us?, Phys. Rev. Lett. **131**, 240001 (2023).
[39] K. Das, S. Lahiri, R. B. Atencia, D. Culcer, and A. Agarwal, Intrinsic nonlinear conductivities induced by the quantum metric, Phys. Rev. B **108**, L201405 (2023).
[40] A. Kirikoshi and S. Hayami, Microscopic mechanism for intrinsic nonlinear anomalous Hall conductivity in noncollinear antiferromagnetic metals, Phys. Rev. B **107**, 155109 (2023).
[41] L.-k. Shi, O. Matsyshyn, J. C. Song, and I. S. Villadiego, Berry-dipole photovoltaic demon and the thermodynamics of photocurrent generation within the optical gap of metals, Phys. Rev. B **107**, 125151 (2023).

[42] N. Wang, D. Kaplan, Z. Zhang, T. Holder, N. Cao, A. Wang, X. Zhou, F. Zhou, Z. Jiang, C. Zhang *et al.*, Quantum-metric-induced nonlinear transport in a topological antiferromagnet, Nature (London) **621**, 487 (2023).
[43] A. Gao, Y.-F. Liu, J.-X. Qiu, B. Ghosh, T. V. Trevisan, Y. Onishi, C. Hu, T. Qian, H.-J. Tien, S.-W. Chen *et al.*, Quantum metric nonlinear Hall effect in a topological antiferromagnetic heterostructure, Science **381**, 181 (2023).
[44] B. Hetényi and P. Lévay, Fluctuations, uncertainty relations, and the geometry of quantum state manifolds, Phys. Rev. A **108**, 032218 (2023).
[45] A. Gao, S.-W. Chen, B. Ghosh, J.-X. Qiu, Y.-F. Liu, Y. Onishi, C. Hu, T. Qian, D. Bérubé, T. Dinh *et al.*, An antiferromagnetic diode effect in even-layered mnbi2te4, Nat. Electron. **7**, 751 (2024).
[46] Y. Wang, Z. Zhang, Z.-G. Zhu, and G. Su, Intrinsic nonlinear ohmic current, Phys. Rev. B **109**, 085419 (2024).
[47] D. Kaplan, T. Holder, and B. Yan, Unification of nonlinear anomalous Hall effect and nonreciprocal magnetoresistance in metals by the quantum geometry, Phys. Rev. Lett. **132**, 026301 (2024).
[48] M. Mehraeen, Quantum kinetic theory of quadratic responses, Phys. Rev. B **110**, 174423 (2024).
[49] T. Liu, X.-B. Qiang, H.-Z. Lu, and X. Xie, Quantum geometry in condensed matter, Natl. Sci. Rev. **12**, nwae334 (2025).
[50] H. Zhu, J. Li, X. Chen, Y. Yu, and Q. Liu, Magnetic geometry induced quantum geometry and nonlinear transports, Nat. Commun. **16**, 4882 (2025).
[51] X.-B. Qiang, T. Liu, Z.-X. Gao, H.-Z. Lu, and X. Xie, A clarification on quantum-metric-induced nonlinear transport, Adv. Sci. **13**, e14818 (2026).
[52] Y. Ulrich, J. Mitscherling, L. Classen, and A. P. Schnyder, Quantum geometric origin of the intrinsic nonlinear Hall effect, Phys. Rev. B **113**, L201107 (2026).
[53] C. Xiao, J. Cao, Q. Niu, and S. A. Yang, Proper definition of intrinsic nonlinear current, Phys. Rev. Lett. **135**, 256306 (2025).
[54] R. Resta, Intrinsic nonlinear Hall effect beyond Bloch geometry, Phys. Rev. B **113**, 125136 (2026).
[55] P. Fontana, V. Velasco, C. Niu, P. D. Ye, P. V. Lopes, K. E. de Souza, M. V. Moutinho, C. Lewenkopf, and M. B. S. Neto, Quantum geometry and the electric magnetochiral anisotropy in noncentrosymmetric polar media, Phys. Rev. Lett. **135**, 106602 (2025).
[56] M. Mehraeen, Quantum response theory and momentum-space gravity, Phys. Rev. Lett. **135**, 156302 (2025).
[57] K. I. Seetharam, C.-E. Bardyn, N. H. Lindner, M. S. Rudner, and G. Refael, Controlled population of Floquet-Bloch states via coupling to Bose and Fermi baths, Phys. Rev. X **5**, 041050 (2015).
[58] T. Morimoto and N. Nagaosa, Topological nature of nonlinear optical effects in solids, Sci. Adv. **2**, e1501524 (2016).
[59] O. Matsyshyn, F. Piazza, R. Moessner, and I. Sodemann, Rabi regime of current rectification in solids, Phys. Rev. Lett. **127**, 126604 (2021).
[60] L. Gerchikov, D. Parshin, and A. Shabaev, Theory of resonance saturation of ir absorption in semiconductors with degenerate resonance bands in electric and magnetic fields, Zh. Eksp. Teor. Fiz. **96**, 1046 (1989).
[61] A.-P. Jauho, N. S. Wingreen, and Y. Meir, Time-dependent transport in interacting and noninteracting resonant-tunneling systems, Phys. Rev. B **50**, 5528 (1994).
[62] K. Johnsen and A.-P. Jauho, Quasienergy spectroscopy of excitons, Phys. Rev. Lett. **83**, 1207 (1999).
[63] S. Kohler, J. Lehmann, and P. Hänggi, Driven quantum transport on the nanoscale, Phys. Rep. **406**, 379 (2005).
[64] A. Kamenev, *Field Theory of Non-Equilibrium Systems* (Cambridge University Press, Cambridge, England, 2011).
[65] See Supplemental Material at http://link.aps.org/supplemental/10.1103/rbck-mt1k for detailed derivations, which includes Refs. [17,26,39,59–64,66].
[66] B. M. Fregoso, Y. Wang, N. Gedik, and V. Galitski, Driven electronic states at the surface of a topological insulator, Phys. Rev. B **88**, 155129 (2013).